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RATIONAL APPROXIMATIONS TO IRRATIONAL COMPLEX NUMBERS*

BY

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We shall call a number $m + ni$, where $i = \sqrt{-1}$, a *complex integer* if m and n are real integers. By a *rational complex fraction* we shall mean a fraction whose numerator and denominator are complex integers. By an *irrational complex number* we shall mean a number not expressible as a rational complex fraction.

Hermite† has proposed a method of forming a suite of rational complex fractions approaching an irrational complex number ω . The method is briefly the following. Consider the positive definite hermitian form

$$(1) \quad F(x, y) = (x - \omega y)(\bar{x} - \bar{\omega}\bar{y}) + k^2 y\bar{y},$$

where \bar{x} , \bar{y} , $\bar{\omega}$ are the conjugate imaginaries of x , y , ω , and where k is real. Let $x = p$, $y = q$ be a pair of complex integers giving a minimum value of the form F (the values $x = 0$, $y = 0$ being excluded). Then the fraction p/q is an approximation to ω satisfying the inequality

$$(2) \quad \left| \omega - \frac{p}{q} \right| \leq \frac{1}{\sqrt{2}qq}.$$

If k be made to decrease from ∞ to 0, the integers p and q change from time to time, and there results an infinite suite of fractions tending toward ω . If ω be a rational complex fraction, a suite is likewise determined, but it terminates with the fraction ω after a finite number of terms.

It is the object of the first part of this paper to investigate the suite of Hermite, to give the fractions a geometrical interpretation, to find a method of calculating them, and to study their properties.‡

The second part of the paper is devoted to approximations by means of

* Part II presented to the Society, Sept. 4, 1916, under the title *Regular continued fractions*.

† *Sur la théorie des formes quadratiques*, Journal für Mathematik, vol. 47 (1854), particularly pp. 357-366.

‡ The principal results of this part have been presented in a brief note in *Comptes Rendus*, vol. 162 (1916), pp. 459-461.

continued fractions. It is shown that the convergents of any continued fraction of the form

$$(3) \quad s_0 \pm \frac{\epsilon_1}{s_1 \pm \frac{\epsilon_2}{s_2 \pm \dots \frac{\epsilon_n}{s_n \pm \dots}}},$$

where $\epsilon_n = \pm 1$ or $\pm i$, and where s_n is a complex integer (such a fraction will be called *regular*), can be derived by a method which is a generalization of the method of Hermite. The properties of the convergents of such a continued fraction, in particular their values as approximations to the sum of the fraction, are investigated, and theorems relative to the periodicity of the continued fraction are established.

I take occasion here to express my indebtedness to Professor Humbert of the Collège de France, who suggested to me a study of the suite of Hermite, and who pointed the way to its geometrical interpretation in his investigation of the analogous problem for the case of reals.*

PART I

THE FRACTIONS OF HERMITE

1. The Group of Picard. The whole theory of rational approximations to complex numbers is concerned with the group of transformations

$$(4) \quad z' = \frac{\alpha z + \beta}{\gamma z + \delta} \quad (\alpha\delta - \beta\gamma = 1)$$

where $\alpha, \beta, \gamma, \delta$ are complex integers. This group is known as the Group of Picard. The group is not properly discontinuous in the z -plane, but if it be defined as a group of space transformations according to the idea of Poincaré† a fundamental polyhedron for the group exists. The space transformations, where ζ is a variable measured along an axis through the origin perpendicular to the z -plane, are

$$(5) \quad \begin{aligned} \rho'^2 &= \frac{\rho^2 \alpha \bar{\alpha} + z \alpha \bar{\beta} + \bar{z} \bar{\alpha} \beta + \beta \bar{\beta}}{\rho^2 \gamma \bar{\gamma} + z \gamma \bar{\delta} + \bar{z} \bar{\gamma} \delta + \delta \bar{\delta}}, \\ z' &= \frac{\rho^2 \alpha \bar{\gamma} + z \alpha \bar{\delta} + \bar{z} \bar{\beta} \bar{\gamma} + \beta \bar{\delta}}{\rho^2 \gamma \bar{\gamma} + z \gamma \bar{\delta} + \bar{z} \bar{\gamma} \delta + \delta \bar{\delta}}, \\ \bar{z}' &= \frac{\rho^2 \bar{\alpha} \gamma + z \gamma \bar{\beta} + \bar{z} \bar{\alpha} \delta + \beta \bar{\delta}}{\rho^2 \gamma \bar{\gamma} + z \gamma \bar{\delta} + \bar{z} \bar{\gamma} \delta + \delta \bar{\delta}}, \end{aligned}$$

where $\rho'^2 = \zeta'^2 + z' \bar{z}'$ and $\rho^2 = \zeta^2 + z \bar{z}$.§ Combining these equations we

* Comptes Rendus, vol. 161 (1915), pp. 717-721; Journal de mathématiques, 7th ser., vol. 2 (1916), pp. 79-103.

† Mémoire sur les groupes kleinéens, Acta Mathematica, vol. 3 (1884), pp. 49-92.

§ Poincaré, loc. cit., p. 54.

get also

$$(6) \quad \zeta' = \frac{\zeta}{\rho^2 \gamma \bar{\gamma} + z \gamma \bar{\delta} + \bar{z} \bar{\gamma} \delta + \delta \bar{\delta}}.$$

For points of the z -plane the transformation (5) is equivalent to (4).

The transformation inverse to (4) is

$$(7) \quad z' = \frac{-\delta z + \beta}{\gamma z - \alpha},$$

hence the inverse space transformation can be written down by replacing $\alpha, \bar{\alpha}, \delta, \bar{\delta}$ by $-\delta, -\bar{\delta}, -\alpha, -\bar{\alpha}$ respectively in (5) and (6).

These space transformations preserve angles both in magnitude and sign, transform spheres into spheres and circles into circles (planes and straight lines being included as special cases of spheres and circles), and transform figures inverse with respect to a sphere into figures inverse with respect to the transformed sphere. These properties are a consequence of the fact that each transformation is equivalent to an even number of inversions in spheres.

The fundamental polyhedron for the Group of Picard has been found by Bianchi* to be the pentahedron lying above the sphere $\xi^2 + \eta^2 + \zeta^2 = 1$ ($z = \xi + i\eta$) and enclosed by the four planes $\xi = 0, \xi = \frac{1}{2}, \eta = -\frac{1}{2}, \eta = \frac{1}{2}$. The transforms of this region by the transformations of the group fill up, without overlapping, the whole space above the z -plane. This division of the half-space can also be derived from the fundamental pentahedron by inverting it in each of its faces, inverting each new pentahedron in each of its faces, and so on ad infinitum.

In what follows we shall find it convenient to adjoin to the fundamental pentahedron that arising from it by reflection in the plane $\xi = 0$. This double pentahedron, which we shall henceforth call the *fundamental pentahedron*, then lies above the sphere $\xi^2 + \eta^2 + \zeta^2 = 1$ and is bounded by the planes $\xi = \pm \frac{1}{2}, \eta = \pm \frac{1}{2}$. The four plane faces we shall call the *lateral faces* of the pentahedron, and the remaining face we shall call its *base*. Of the five vertices, that at ∞ we shall call the *peak*, reserving for the other four the name *vertices*. The vertices lie in the plane $\zeta = \frac{1}{2}\sqrt{2}$.

If a transformation of the group be applied to this pentahedron there results a pentahedron bounded by spheres orthogonal to the z -plane, and we shall apply to the faces and vertices of this pentahedron the names of the corresponding parts in the fundamental pentahedron. Thus the four lateral faces meet in a point, the peak, in the z -plane. The four vertices and the base lie in the space above the z -plane.

The repetitions of the fundamental pentahedron by the translations of the group $z' = z + \beta$ fill the whole space outside the spheres of unit radius

* *Mathematische Annalen*, vol. 38 (1891), pp. 313-333.

with centers at the points of the z -plane whose z -coördinates are complex integers. Each of these pentahedra has its peak at ∞ and its vertices in the plane $\zeta = \frac{1}{2}\sqrt{2}$.

The transformation $z' = -1/z$ carries the fundamental pentahedron into another with the same base and with its peak at the origin. By the transformations $z' = z + \beta$ this pentahedron is carried into others with bases in common with those of the preceding paragraph and with peaks at the points of the z -plane whose coördinates are complex integers.

2. The positive definite hermitian form. The binary form

$$(8) \quad \phi(x, y) = ax\bar{x} + bx\bar{y} + \bar{b}x\bar{y} + cy\bar{y},$$

where a and c are real, is called a hermitian form. It is definite if the determinant, $D = b\bar{b} - ac$, is negative. We shall suppose that $a > 0$; that is, that the form is positive. If we set $x/y = z$ and equate (8) to zero, we obtain the equation of an imaginary circle in the z -plane,

$$(9) \quad az\bar{z} + bz + \bar{b}\bar{z} + c = 0.$$

The one-parameter family of spheres through this circle contains two real point spheres; these are, if we write $b = b_1 + ib_2$, at the points

$$(10) \quad \xi = -\frac{b_1}{a}, \quad \eta = \frac{b_2}{a}, \quad \zeta = \pm \frac{\sqrt{-D}}{a}.$$

The point lying in the upper half space ($\zeta = +\sqrt{-D}/a$) is customarily chosen as the *representative point of the form*.^{*} This point has the property that if x and y be replaced by the new variables

$$X = \alpha x + \beta y, \quad Y = \gamma x + \delta y \quad (\alpha\delta - \beta\gamma = m \neq 0)$$

where m is real, the representative point of the new form is derived from that of the old by the space transformation corresponding to

$$Z = \frac{\alpha z + \beta}{\gamma z + \delta}.$$

In the form (1) of the Introduction $D = -k^2$. Writing $\omega = \omega_1 + i\omega_2$ we find the representative point of the form to be

$$\xi = \omega_1, \quad \eta = \omega_2, \quad \zeta = +\sqrt{k^2},$$

or

$$(11) \quad z = \omega, \quad \zeta = k,$$

where k is the positive square root of k^2 . As k decreases from ∞ to 0, the representative point traces the half of the line $z = \omega$ lying in the upper half space, passing from ∞ to the z -plane.

^{*} Fricke-Klein, *Theorie der automorphen Functionen*, vol. 1, p. 455.

Minimum of the form. As a step toward finding the minimum of the form (8)—that is, the minimum for complex integral values of x and y ($x = 0$, $y = 0$ excluded)—we shall establish the following proposition.

If the representative point of the form lies in a pentahedron whose peak is at infinity the minimum of the form is the first coefficient.

The first coefficient is a permissible value, being given by $x = 1$, $y = 0$. Obviously $x = p$, $y = 0$ where $|p| > 1$ yields a greater value. We shall treat two cases: $x = p$, $y = 1$, and $x = p$, $y = q$, where $|q| > 1$. Thus all possible values of x and y are considered.

The representative point of the form lies outside the unit sphere whose center is $z = p$; that is, outside the sphere

$$(z - p)(\bar{z} - \bar{p}) + \zeta^2 = 1.$$

The coördinates of the representative point from (10) are

$$z = (-b_1 + ib_2)/a = -\bar{b}/a, \quad \zeta = \sqrt{ac - b\bar{b}}/a,$$

and these then satisfy the inequality

$$\left(-\frac{\bar{b}}{a} - p\right)\left(-\frac{b}{a} - \bar{p}\right) + \frac{ac - b\bar{b}}{a^2} \geq 1,$$

whence

$$(12) \quad ap\bar{p} + bp + \bar{b}\bar{p} + c \geq a.$$

That is, $\phi(p, 1) \geq a$, whatever the integer p may be.

The representative point also lies above the plane $\zeta = \frac{1}{2}\sqrt{2}$. Then

$$\frac{ac - b\bar{b}}{a^2} \geq \frac{1}{2}.$$

But

$$\begin{aligned} \phi(p, q) &= ap\bar{p} + bp\bar{q} + \bar{b}\bar{p}q + cq\bar{q} \\ (13) \quad &= a\left[\left(p + \frac{\bar{b}q}{a}\right)\left(\bar{p} + \frac{b\bar{q}}{a}\right) + \frac{ac - b\bar{b}}{a^2}q\bar{q}\right] \\ &\geq \frac{aq\bar{q}}{2} \geq a, \end{aligned}$$

since $q\bar{q} \geq 2$. The proposition is thus established.

Suppose now that the representative point does not lie in a pentahedron whose peak is at ∞ . We observe that the set of values of the form given by complex integral values of the variables is unchanged if we make the substitution

$$(14) \quad x = \alpha X + \beta Y, \quad y = \gamma X + \delta Y, \quad \alpha\delta - \beta\gamma = 1,$$

$\alpha, \beta, \gamma, \delta$ being complex integers. For complex integral values of X and Y

yield complex integral values of x and y , and conversely. The new form in X and Y will have the same minimum as the original.

Now let

$$(15) \quad z = \frac{\alpha Z + \beta}{\gamma Z + \delta}$$

be the transformation of the Group of Picard carrying the fundamental pentahedron into the pentahedron in which the representative point of the form (8) lies; and let the coefficients $\alpha, \beta, \gamma, \delta$ of (15) be used in the substitution (14). Then the representative point of the new form lies within the fundamental pentahedron, and its minimum is the first coefficient. The new form we find, on making the substitution (14), to be

$$(16) \quad \phi'(X, Y) = (a\alpha\bar{\alpha} + b\alpha\bar{\gamma} + \bar{b}\alpha\bar{\gamma} + c\gamma\bar{\gamma})X\bar{X} + \dots$$

The first coefficient is then $\phi(\alpha, \gamma)$.

The peak of the pentahedron in which the representative point of the form $\phi(x, y)$ lies is the transform of $Z = \infty$; that is, $z = \alpha/\gamma$. We have thus proved the first part of the following theorem:

THEOREM. *If α/γ (in its lowest terms) is the z -coördinate of the peak of the pentahedron in which the representative point of the form $\phi(x, y)$ lies, the minimum of the form is $\phi(\alpha, \gamma)$.*

Conversely, if $\phi(\alpha, \gamma)$ is the minimum of the form the peak of the pentahedron in which its representative point lies has the z -coördinate α/γ .

The converse follows at once in case the form attains its minimum for a single pair of variables. We see from the derivation of (12) and (13) that the minimum of the form is given by a single pair of values in general. Now (12) becomes an equality only if the representative point lies in a base common to two pentahedra and (13) becomes an equality if the representative point lies in a vertex, where several pentahedra meet. In these exceptional cases there are two or more pairs of values giving the minimum, but we easily show that this arises from the fact that the representative point may be said to belong to two or more pentahedra.

3. Geometrical interpretation of the fractions of Hermite. The representative point of the form (1) of the Introduction we found (11) to be $z = \omega$, $\zeta = k$. The fraction of Hermite for a given value of k is p/q where $x = p$, $y = q$ gives the minimum of the form; that is, it is the z -coördinate of the peak of the pentahedron in which the representative point of the form lies. As k decreases from ∞ to 0 the representative point traces the half-line $z = \omega$ from ∞ to the z -plane. We have then the following interpretation:

Let a moving point trace the half of the line $z = \omega$ lying in the upper half-space, passing from ∞ to the z -plane. The z -coördinates of the peaks of the successive pentahedra through which this point passes are the fractions of Hermite tending toward the value ω .

From this interpretation it is obvious that there are only a finite number of fractions if ω is rational. The moving point eventually enters and remains within pentahedra with peaks at ω , and the suite terminates.

4. The S - and Σ -spheres. In this section we shall deduce certain geometrical facts concerning the pentahedral division of the half-space which will be frequently used in the following pages.

A plane, $\zeta = h$, parallel to the z -plane is transformed by a transformation of the group, $z' = (\alpha z + \beta)/(\gamma z + \delta)$, into a sphere tangent to the z -plane. We get its equation from the inverse of equation (6). It is

$$h = \frac{\zeta}{\rho^2 \gamma \bar{\gamma} - z \gamma \bar{\alpha} - \bar{z} \bar{\gamma} \alpha + \alpha \bar{\alpha}}.$$

Since $\rho^2 = z\bar{z} + \zeta^2$, this can be written

$$(17) \quad \left(z - \frac{\alpha}{\gamma}\right) \left(\bar{z} - \frac{\bar{\alpha}}{\bar{\gamma}}\right) + \left(\zeta - \frac{1}{2h\gamma\bar{\gamma}}\right)^2 = \frac{1}{4h^2 \gamma^2 \bar{\gamma}^2}.$$

From this we see that the center of the sphere is the point $z = \alpha/\gamma$, $\zeta = 1/2h\gamma\bar{\gamma}$, and that its radius is $1/2h\gamma\bar{\gamma}$. The sphere is thus tangent to the z -plane at the point $z = \alpha/\gamma$.

There are two planes of particular importance of the form $\zeta = h$. One of these is $\zeta = \frac{1}{2}\sqrt{2}$, which contains the vertices of all pentahedra with peaks at ∞ . By the transformation mentioned above this becomes a sphere of radius $1/\sqrt{2}\gamma\bar{\gamma}$ tangent to the z -plane at $z = \alpha/\gamma$. We shall call this sphere $\Sigma(\alpha/\gamma)$. The transformation in question carries $z = \infty$ to $z' = \alpha/\gamma$. The part of the plane above $\zeta = \frac{1}{2}\sqrt{2}$ is carried into the interior of $\Sigma(\alpha/\gamma)$. Hence *all the pentahedra with peaks at $z = \alpha/\gamma$ lie within the sphere $\Sigma(\alpha/\gamma)$ and their vertices lie upon its surface.*

A second important plane is $\zeta = 1$, which is tangent to the bases of all the pentahedra with peaks at ∞ . It is transformed into a sphere of radius $1/2\gamma\bar{\gamma}$ tangent to the z -plane at α/γ . This sphere we shall call $S(\alpha/\gamma)$. We see then that *the bases of all pentahedra with peaks at $z = \alpha/\gamma$ are tangent to the sphere $S(\alpha/\gamma)$.*

It is well known that if p and q are any two complex integers without a common factor, other two complex integers, β and δ , can be found such that $p\delta - \beta q = 1$. Then $z' = (pz + \beta)/(qz + \delta)$ is a transformation of the Group of Picard which carries $z = \infty$ to $z' = p/q$. Hence, *each rational point, p/q , of the z -plane is a peak for infinitely many pentahedra. The vertices of these pentahedra lie upon the surface of $\Sigma(p/q)$ and their bases are tangent to $S(p/q)$.*

We shall find it convenient to represent the point ∞ by $1/0$. For the sake of uniformity we shall designate the planes $\zeta = 1$ and $\zeta = \frac{1}{2}\sqrt{2}$ by

$S(1/0)$ and $\Sigma(1/0)$ respectively. The space above these planes corresponds to the interiors of the other S - or Σ -spheres.

Consider now $S(p/q)$. Its diameter is $1/q\bar{q}$. The sphere $S(p/q)$ will lie below $\zeta = 1$ in general, but will be tangent if $q\bar{q} = 1$. In this latter case $p/q = s$, a complex integer, and we know that a pentahedron with peak at s and another with peak at ∞ have a base in common. The point of tangency lies in the common base. Since $\zeta = 1$ can be transformed into any other S -sphere we have the result:

The sphere $S(p'/q')$ will be tangent to the sphere $S(p/q)$ if a pentahedron with peak at p'/q' has a base in common with a pentahedron with peak at p/q ; in all other cases the spheres are entirely exterior to one another.

The diameter of $\Sigma(p/q)$ is $\sqrt{2}/q\bar{q}$. If $q\bar{q} = 1$, so that $p/q = s/1$, the center is at $z = s$, $\zeta = 1/\sqrt{2}$. The center thus lies in the plane $\zeta = \frac{1}{2}\sqrt{2}$, and $\Sigma(s/1)$ and $\Sigma(1/0)$ cut orthogonally. Again, if $q\bar{q} = 2$ (that is, $q = \pm 1 \pm i$) the diameter of $\Sigma(p/q)$ is $\frac{1}{2}\sqrt{2}$, and $\Sigma(p/q)$ touches $\Sigma(1/0)$. By a simple division we can put p/q in the form $s + 1/(1-i)$, or $s + (1+i)/2$, where s is an integer. The point of tangency is $z = s + (1+i)/2$, $\zeta = \frac{1}{2}\sqrt{2}$, a vertex of certain of the pentahedra with peaks at ∞ . It follows that in this case certain pentahedra with peaks at p/q must also have vertices at this point of tangency. If $q\bar{q} > 2$, $\Sigma(p/q)$ lies entirely below $\zeta = \frac{1}{2}\sqrt{2}$. Since $\zeta = \frac{1}{2}\sqrt{2}$ can be transformed into any other Σ -sphere we have the result:

The sphere $\Sigma(p'/q')$ will intersect the sphere $\Sigma(p/q)$ orthogonally if a pentahedron with peak at p'/q' and a pentahedron with peak at p/q have a common base; $\Sigma(p'/q')$ will touch $\Sigma(p/q)$ if a pentahedron with peak at p'/q' and a pentahedron with peak at p/q have a common vertex but no common base; the two Σ -spheres will be entirely exterior to one another in all other cases.

The analytical condition for a common base is expressed in the following

THEOREM. *The necessary and sufficient condition that a pentahedron with peak at p'/q' and a pentahedron with peak at p/q (the fractions being in their lowest terms) have a common base is that $pq' - p'q = \pm 1$ or $\pm i$.*

If two such pentahedra have a common base, let $z' = (\alpha z + \beta)/(\gamma z + \delta)$ be the transformation of the Group of Picard carrying the fundamental pentahedron into the one with peak at p/q . Then the pentahedron with peak at the origin and with the same base as the fundamental pentahedron is carried into the one with peak at p'/q' . But $z = \infty$ and $z = 0$ become respectively α/γ and β/δ , and the relation $\alpha\delta - \beta\gamma = 1$ shows that α and γ , also β and δ , are without a common factor. Since $p/q = \alpha/\gamma$, $p'/q' = \beta/\delta$, we have $p = \epsilon\alpha$, $q = \epsilon\gamma$, $p' = \eta\beta$, $q' = \eta\delta$, where ϵ and η are of the form ± 1 or $\pm i$. Then

$$pq' - p'q = \epsilon\eta(\alpha\delta - \beta\gamma) = \epsilon\eta = \pm 1 \quad \text{or} \quad \pm i.$$

The condition is thus necessary.

It is also sufficient. For if $pq' - p'q = \pm 1$ or $\pm i$ the transformation $z' = (pz + \epsilon p')/(qz + \epsilon q')$, where $\epsilon = \pm 1$ or $\mp i$ respectively, has determinant 1 and is thus of the Group of Picard. It transforms ∞ and 0 into p/q and p'/q' respectively. Now a pentahedron with peak at ∞ and one with peak at 0 have a common base, and these are carried into two with a common base and with peaks at p/q and p'/q' . The theorem is thus established.

5. The calculation of the fractions of Hermite. According to the interpretation (Section 3) the calculation of the fractions reduces to the problem of finding the z -coordinates of the successive peaks of the pentahedra through which the line $z = \omega$ passes. We observe that two pentahedra with a common lateral face have the same peak; hence a moving point on the line $z = \omega$ can pass from one pentahedron into another with a different peak only by passing through the common base of the two pentahedra.* An application of the theorem at the close of the preceding section gives the following result:

If p'/q' and p/q are successive fractions of Hermite in the development of a number ω , $pq' - p'q = \pm 1$ or $\pm i$.†

By multiplying p and q by a suitable one of the numbers ± 1 , $\pm i$, we can have $pq' - p'q$ equal always to any one of the four quantities desired. The method of calculation to be given yields always $pq' - p'q = -1$, where p/q is the fraction following p'/q' .

The point $z = \omega$, $\zeta = k$ lies in a pentahedron with peak at ∞ if k is large. The first fraction is then $1/0$, whatever the value of ω . The pentahedra with bases in common with those with peaks at ∞ have their peaks at the complex integers, and the common bases project orthogonally upon the z -plane into unit squares whose sides are parallel to the coordinate axes and whose centers are the integers. The second peak is then at a , the nearest complex integer to ω ; that is, an integer such that the real part and the imaginary part of $\omega - a$ are each less than or equal to $\frac{1}{2}$ in absolute value. Writing the second fraction $a/1$ we have $a \times 0 - 1 \times 1 = -1$.

We set now the following problem:

Given two successive fractions of Hermite, p'/q' and p/q , in the development of a number ω , where $pq' - p'q = -1$, to find the following fraction, P/Q .

The geometrical interpretation of the problem is shown in Figure 1, p. 10.

* If the line passes through a vertex of the first pentahedron the moving point may pass into a pentahedron having only a vertex in common with the first. But the vertex belongs also to other pentahedra, one of which has a base in common with the two preceding and can be taken as intermediate between them.

This case does not arise if ω is irrational, the z -coordinate of a vertex being always rational. For the vertex is the transform of one of the vertices of the fundamental pentahedron, $z = (\pm 1 \pm i)/2$, $\zeta = \frac{1}{2}\sqrt{2}$, and we see from the second of equations (5) that with these values of z and ζ , z' is rational.

† This result appears in Hermite's memoir.

As k decreases the moving point in the line $z = \omega$ passes from a pentahedron D_1 with peak at p'/q' , traversing its base, and entering a pentahedron D_2 with

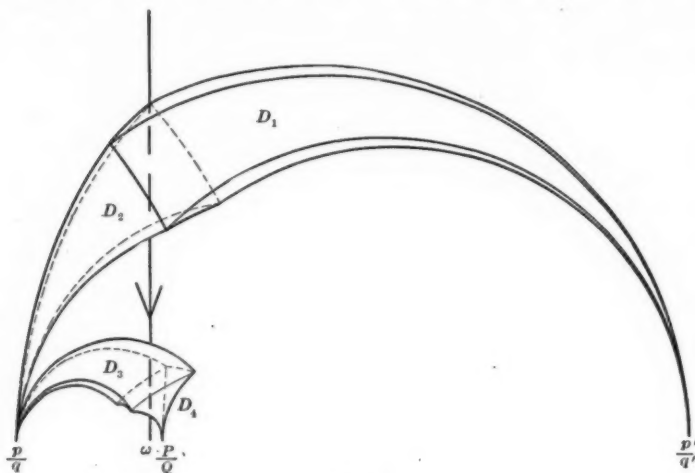


FIG. 1

peak at p/q . After passing through a certain number of pentahedra with peaks at p/q it passes from one of them, D_3 , by its base and enters a pentahedron, D_4 , with its peak at P/Q , the fraction sought.

Let us apply to this configuration the transformation

$$(18) \quad Z = \frac{q'z - p'}{qz - p},$$

which, since its determinant is 1, is of the Group of Picard. The transformed configuration is shown in Figure 2, D'_1, D'_2, D'_3, D'_4 being the transforms of

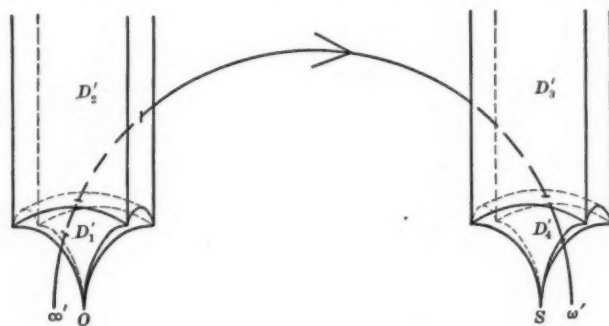


FIG. 2

D_1, D_2, D_3, D_4 . The fractions p/q and p'/q' become respectively ∞ and 0. The pentahedron D'_2 is then the fundamental pentahedron, and D'_1 is the pentahedron with the same base. The pentahedron D'_3 also has its peak at ∞ , and D'_4 , which has the same base, has its peak at a complex integer s , to be found.

Writing ∞' and ω' for the transforms of ∞ and ω , we have

$$(19) \quad \infty' = \frac{q'}{q}, \quad \omega' = \frac{q' \omega - p'}{q \omega - p} = \frac{q'}{q} - \frac{1}{u},$$

where $u = q(q\omega - p)$. The half-line $z = \omega$ is transformed into the semi-circle C orthogonal to the z -plane through these two points. (This follows from the fact that circles are transformed into circles and that angles are preserved.) In tracing this semicircle from ∞' to ω' we enter the pentahedra with peaks at ∞ by the base of D'_2 and leave them by the base of D'_3 .

The circle C has its center in the z -plane at $z = \frac{1}{2}(\infty' + \omega') = q'/q - 1/2u$, and its radius is $\frac{1}{2}|\infty' - \omega'|$, or $1/|2u|$. Its equations can be written

$$(20) \quad \left(z - \frac{q'}{q} + \frac{1}{2u}\right) \left(\bar{z} - \frac{\bar{q}'}{q} + \frac{1}{2\bar{u}}\right) + \zeta^2 = \frac{1}{4u\bar{u}},$$

$$z - \frac{q'}{q} + \frac{1}{2u} = \frac{t}{u},$$

where t is a real parameter. The first of these equations is the sphere with the same center and radius as C ; the second is the plane through ∞' and ω' orthogonal to the z -plane. The parameter t is positive for points between ∞' and the highest point of C and negative for points between the highest point and ω' .

The base common to D'_3 and D'_4 is the unit sphere with center at the point $z = s$ in the z -plane,

$$(21) \quad (z - s)(\bar{z} - \bar{s}) + \zeta^2 = 1.$$

The z -coördinate of the point of intersection of the circle (20) with the sphere (21) we find after an easy calculation to be

$$(22) \quad z_i = \frac{q'}{q} - \frac{1}{2u} + \frac{1}{u} \left[\frac{-\left(\frac{q'}{q} - \frac{1}{2u} - s\right) \left(\frac{\bar{q}'}{q} - \frac{1}{2\bar{u}} - \bar{s}\right) - \frac{1}{4u\bar{u}} + 1}{\frac{1}{u} \left(\frac{\bar{q}'}{q} - \frac{1}{2\bar{u}} - \bar{s}\right) + \frac{1}{\bar{u}} \left(\frac{q'}{q} - \frac{1}{2u} - s\right)} \right].$$

In order that the point of intersection lie in the base of D'_3 , z_i must lie within the unit square whose center is s and whose sides are parallel to the coördinate axes. This condition is also sufficient to determine s if we rule out the case $s = 0$, which is the peak of D'_1 . Otherwise stated, s is the complex integer

different from zero such that $z_i - s$ has a real and an imaginary part each less than or equal to $\frac{1}{2}$ in absolute value. Putting $U = q'/q - 1/2u$, we can write $z_i - s$ in the following form

$$(23) \quad U - s = \frac{u\bar{u}(U - s)(\bar{U} - \bar{s}) - u\bar{u} + \frac{1}{4}}{u(uU + \bar{u}\bar{U})}.$$

Having found s to satisfy the above condition we can transform back to the original configuration by means of (18), i. e., $z = (pZ - p')/(qZ - q')$. Hence we have

$$(24) \quad \frac{P}{Q} = \frac{ps - p'}{qs - q'}$$

and we can take

$$(25) \quad P = ps - p', \quad Q = qs - q'.$$

We note also that $Pq - pQ = -1$.

The determination of s from the expression (23) is not practicable, and a method will now be devised in which it will be necessary to have recourse to (23) only in certain doubtful cases.

The z -coördinate of the point of intersection of the semicircle C with the plane $\zeta = \frac{1}{2}\sqrt{2}$ (which passes through the vertices of the pentahedra with peaks at ∞) will differ but little from that of the intersection with the base sought. Combining the equations of (20) and putting $\zeta^2 = \frac{1}{2}$ we have

$$\frac{t^2}{u\bar{u}} + \frac{1}{2} = \frac{1}{4u\bar{u}},$$

whence

$$t = \pm \frac{1}{2}\sqrt{1 - 2u\bar{u}}.$$

The negative value of t gives the intersection nearest ω' . From the second of equations (20) we have for z_0 , the intersection sought,

$$(26) \quad z_0 = \frac{q'}{q} - \frac{1}{2u}(1 + \sqrt{1 - 2u\bar{u}}).$$

Let s_0 be the integer nearest z_0 . Under what conditions is s_0 the integer s of condition (23)? The semicircle C either intersects the base B lying above z_0 , or intersects an adjoining base and passes under one of the sides bounding the base before reaching the plane $\zeta = \frac{1}{2}\sqrt{2}$.* Hence s is one of the integers s_0, s_1, s_2 , where s_1 and s_2 are the peaks of two of the pentahedra with bases adjoining B . Since C passes through the base above the origin before arriving at the bases just mentioned, $|s_1|$ and $|s_2|$ are less than $|s_0|$. Also $|s_1 - s_0| = 1$ and $|s_2 - s_0| = 1$, since the bases above s_1 and s_2 adjoin B . (That is, one has

* This latter possibility renders the calculation of the fractions a more complicated procedure than in the case of reals.

a real part, the other an imaginary part, differing by one unit from that of s_0 .) Now the semicircle C can intersect each sphere with center in the z -plane but once. Hence if C intersects the base above s_1 , say, the later point of intersection of C with the plane $\zeta = \frac{1}{2}\sqrt{2}$ must still be within the unit sphere with center s_1 , which forms that base. We find easily that $\zeta = \frac{1}{2}\sqrt{2}$ intersects this sphere in a circle of radius $\frac{1}{2}\sqrt{2}$. Hence if $(z_0 - s_1)(\bar{z}_0 - \bar{s}_1) \geq \frac{1}{2}$, C cannot have intersected the base above s_1 .

We can now state the following rule of calculation, which would appear to be in as convenient a form as possible for actual use.

Let s_0 be the nearest complex integer to the number z_0 , where

$$(27) \quad z_0 = \frac{q'}{q} - \frac{1}{2u}(1 + \sqrt{1 - 2u\bar{u}}).$$

Determine the complex integer s as follows. Let s_1, s_2 be the two (in general two) complex integers such that

$$(28) \quad |s_0 - s_i| = 1, \quad |s_i| < |s_0| \quad (i = 1, 2).$$

If neither of the inequalities

$$(29) \quad (z_0 - s_i)(\bar{z}_0 - \bar{s}_i) < \frac{1}{2} \quad (i = 1, 2),$$

is satisfied, $s = s_0$.

If the inequality is satisfied for $i = 1$ (or 2),* substitute s_0 into the expression (23). If this expression has a real and an imaginary part each less than or equal to $\frac{1}{2}$ in absolute value, $s = s_0$; if not, $s = s_1$ (or s_2).

Then

$$(30) \quad P = ps - p', \quad Q = qs - q'.$$

Beginning with the two fractions $1/0$ and $a/1$ and applying this method of procedure, we can calculate step by step the fractions of the suite.

In applying this process it is not necessary in the majority of cases to have recourse to condition (23). If such recourse is necessary there is recompense sometimes in the fact that P_1/Q_1 , the fraction following P/Q , can be found at once. For if $s = s_j$ ($j = 1$ or 2) in the preceding rule, we know that the next peak determined by C is at s_0 , for the point of intersection of C with the plane $\zeta = \frac{1}{2}\sqrt{2}$ certainly lies in a pentahedron with peak at s_0 . Therefore, if $P/Q = (ps_i - p')/(qs_i - q')$, we have $P_1/Q_1 = (ps_0 - p')/(qs_0 - q')$. This yields the result

$$(31) \quad P_1 = \frac{ps_0 - p'}{s_i - s_0}, \quad Q_1 = \frac{qs_0 - q'}{s_i - s_0},$$

the denominators, which are of the form $\pm 1, \pm i$, being introduced in order to make $P_1 Q - P Q_1 = -1$.

* The inequality cannot be satisfied for both s_1 and s_2 , since $|s_1 - s_2| = \sqrt{2}$.

6. The modified hermitian suite. The preceding rule of calculation gives all of the fractions of Hermite in the development of a number ω . We get a suite of fractions differing little from the true suite if in the preceding process we take always $s = s_0$. We thus dispense altogether with condition (23), and find P and Q directly from (27) and (30). As we have just seen this usually gives the fraction immediately following p/q , but in some cases it yields the *second* fraction after p/q . This modified suite thus differs from the original by the omission of certain fractions, but no two successive fractions of the original suite are omitted.

It is easily seen that a fraction can be omitted only if the line $z = \omega$ passes through but *two* pentahedra with peaks at the fraction, but that it is not always omitted under these circumstances.

7. Properties of the fractions of Hermite. Let p/q be a fraction in the suite determined by the number ω . Then the line $z = \omega$ intersects pentahedra with peaks at p/q . As these pentahedra all lie within the sphere $\Sigma(p/q)$, which has the radius $1/\sqrt{2q\bar{q}}$, we have at once the inequality (2) of the Introduction*

$$(32) \quad \left| \omega - \frac{p}{q} \right| \leq \frac{1}{\sqrt{2q\bar{q}}}.$$

Since p/q is the minimum of the form (1) for a certain value of k the representative point of the form, $z = \omega$, $\zeta = k$, lies in a pentahedron with peak at p/q , and therefore lies in $\Sigma(p/q)$. This gives the condition

$$\left(\omega - \frac{p}{q} \right) \left(\bar{\omega} - \frac{\bar{p}}{\bar{q}} \right) + \left(k - \frac{1}{\sqrt{2q\bar{q}}} \right)^2 \leq \left(\frac{1}{\sqrt{2q\bar{q}}} \right)^2,$$

or

$$(33) \quad (p - \omega q)(\bar{p} - \bar{\omega}\bar{q}) + k^2 q\bar{q} \leq k\sqrt{2}.$$

This is the inequality from which Hermite, in the memoir cited, derives his results, and we thus see its geometrical meaning.

We turn now to a comparison of p/q with the fraction P/Q which immediately follows it in the suite.

Comparison of Q and q . Let M be the sphere forming the common base of the two pentahedra with peaks at p/q and P/Q . The line $z = \omega$, traced from ∞ to the z -plane, enters this sphere through the common base, passing out of a pentahedron with peak at p/q and entering one with peak at P/Q . The point p/q of the z -plane is outside M while P/Q and also ω are inside. Now let us consider $S(p/q)$ and $S(P/Q)$, which are tangent to the bases of all pentahedra with peaks at p/q and P/Q respectively. Since the one

* If ω is irrational the sign of equality can be removed. For the only points of the pentahedra on the equator of $\Sigma(p/q)$ are vertices, and these have rational z -coordinates. A similar remark is applicable to the next inequality. See Note, Section 5.

set of pentahedra can be derived from the other by an inversion in the common base, it follows that $S(p/q)$ and $S(P/Q)$ are inverse spheres with respect to M . The former lies outside M and the latter inside, and they are tangent at a point in the common base. The former sphere is therefore larger than the latter. Using the values of the radii found in Section 4, we have $1/2q\bar{q} > 1/2Q\bar{Q}$; or

$$(34) \quad |Q| > |q|.$$

The convergence of the suite to the value ω now follows at once. If p_n/q_n is written as the general term of the suite, (34) shows that $|q_n|$ approaches infinity as n increases without limit. The convergence then follows from (32).

In order to examine more closely into the relation between Q and q let us find the equation of the sphere M . Any base is the transform of the base of the fundamental pentahedron, $z\bar{z} + \zeta^2 = 1$, or $\rho^2 = 1$, by some transformation of the group, $z' = (\alpha z + \beta)/(\gamma z + \delta)$. We get the transform at once from the first of equations (5), using the inverse transformation,

$$1 = \frac{(z\bar{z} + \zeta^2)\delta\bar{\delta} - z\delta\bar{\beta} - \bar{z}\delta\beta + \beta\bar{\beta}}{(z\bar{z} + \zeta^2)\gamma\bar{\gamma} - z\gamma\bar{\alpha} - \bar{z}\gamma\alpha + \alpha\bar{\alpha}}.$$

This can be written in the following form:

$$(35) \quad \left(z - \frac{\beta\bar{\delta} - \alpha\bar{\gamma}}{\delta\bar{\delta} - \gamma\bar{\gamma}}\right) \left(\bar{z} - \frac{\bar{\beta}\delta - \bar{\alpha}\gamma}{\delta\bar{\delta} - \gamma\bar{\gamma}}\right) + \zeta^2 = \frac{1}{(\delta\bar{\delta} - \gamma\bar{\gamma})^2}.$$

We see that this sphere has its center at the point $z = (\beta\bar{\delta} - \alpha\bar{\gamma})/(\delta\bar{\delta} - \gamma\bar{\gamma})$ and that its radius is $1/|\delta\bar{\delta} - \gamma\bar{\gamma}|$.

Now the transformation carrying the base of the fundamental pentahedron into the base common to two pentahedra with peaks at p/q and P/Q is $z' = (Pz - p)/(Qz - q)$. Hence we find that the center of M is

$$z = (P\bar{Q} - p\bar{q})/(Q\bar{Q} - q\bar{q})$$

and its radius is $1/(Q\bar{Q} - q\bar{q})$. This may be stated as follows:

If R is the radius of the sphere forming the common base between pentahedra with peaks at two successive fractions of Hermite, p/q and P/Q , Q and q satisfy the equation

$$(36) \quad Q\bar{Q} = q\bar{q} + 1/R.$$

Representing the successive fractions in the suite by $p_0/q_0, p_1/q_1, \dots, p_n/q_n, \dots$, we shall now prove the following:

Writing $q_{n+1}q_{n+1} = q_nq_n + N_n$, then

$$(37) \quad \lim_{n \rightarrow \infty} N_n = \infty.$$

To prove this it is sufficient, since N_n is an integer, to establish that there

are only a finite number of values of n yielding a given value N of N_n ; that is, that the line $z = \omega$ can pass through only a finite number of spheres of the system forming bases whose radii are $1/N$. The point ω is within all these spheres, hence the spheres intersect. A reference to the fundamental pentahedron shows that two spheres of the system dividing the half space into pentahedra intersect always at an angle $\pi/3$ or $\pi/2$. The equators of the spheres in the z -plane are equal circles mutually intersecting at these angles, and enclosing the point ω . Obviously only a finite number of such circles exist, and the theorem is established. [A simple geometrical construction shows that not more than four equal circles enclosing a point and mutually intersecting at one or other of the given angles exist; hence not more than four values of N_n are equal.]

Comparison of P and p . If we make the transformation $z' = 1/z$ (which belongs to the group, for in writing it $z' = i/iz$ it has the determinant 1), p/q and P/Q are transformed respectively into q/p and Q/P . The spheres $S(p/q)$ and $S(P/Q)$ become $S(q/p)$ and $S(Q/P)$ respectively. These latter spheres are tangent to and inverse with respect to M' , the transform of M , one lying within M' and the other without.

The transformation $z' = 1/z$ in the plane is geometrically equivalent to an inversion in the unit circle $z\bar{z} = 1$ followed by a reflection in the real axis. Then the corresponding space transformation is equivalent to an inversion in the unit sphere $z\bar{z} + \zeta^2 = 1$ followed by a reflection in the plane $\eta = 0$. The latter transformation has no effect upon the dimensions of the figure.

We consider three cases according as (a) the origin is without M , (b) the origin is within M , or (c) the origin lies upon the surface of M .

(a) If the origin is outside M the inversion in the sphere $z\bar{z} + \zeta^2 = 1$, whose center is the origin, followed by the reflection mentioned above, transforms the interior and exterior of M into the interior and exterior respectively of M' . The sphere $S(Q/P)$ is then inside M' and $S(q/p)$ is outside, and the latter is thus the larger. Their radii then give the inequality $1/2p\bar{p} > 1/2P\bar{P}$, or

$$(38) \quad |P| > |p|.$$

(b) The interior of a sphere enclosing the pole of inversion is transformed into the exterior of the inverse sphere; hence in this case the interior and exterior of M become the exterior and interior respectively of M' . The sphere $S(Q/P)$ is then outside M' and $S(q/p)$ is inside, and the former is the larger. Whence $1/2P\bar{P} > 1/2p\bar{p}$, or

$$(39) \quad |P| < |p|.$$

Now the only sphere of the system dividing the half-space into pentahedra which has the origin on its interior is the unit sphere $z\bar{z} + \zeta^2 = 1$ itself. The

surface of this sphere (as of any sphere of the system) is divided by the other spheres of the system into infinitely many regions. Of these regions an infinite number are bases of pentahedra and an infinite number are lateral faces. If the line $z = \omega$ intersects the sphere in one of the bases there is a fraction P/Q of the suite for which $|P| < |p|$. Since the line can intersect the sphere but once there can not be more than one such fraction. If $|\omega| \geq 1$ there is obviously none.

The points P/Q and p/q are inverse with respect to $z\bar{z} = 1$, the equator of M . Writing the equation of the inversion $z' = 1/\bar{z}$ we have $P/Q = \bar{q}/\bar{p}$. As the fractions in this equation are in their lowest terms $P = \epsilon\bar{q}$, $Q = \epsilon\bar{p}$, where $\epsilon = \pm 1$ or $\pm i$. The relation $Pq - pQ = -1$ gives at once the following results:

$$(40) \quad P = \bar{q}, \quad Q = \bar{p}, \quad P\bar{P} = p\bar{p} - 1, \quad Q\bar{Q} = q\bar{q} + 1.$$

We shall now show that this exceptional fraction can be as far from the beginning of the suite as is desired.

Given an integer n however large, a number ω can be found such that in its hermitian suite, $p_0/q_0, p_1/q_1, \dots, |p_{m+1}| < |p_m|$, where m is some integer greater than n .

Let ω' be an irrational number on the circumference of the circle $z\bar{z} = 1$, and let $p_0/q_0, \dots, p_n/q_n, \dots$ be its suite of Hermite, which contains an infinite number of fractions. The line $z = \omega'$ intersects only pentahedra outside the sphere $z\bar{z} + \zeta^2 = 1$. Now ω' can be varied slightly in any direction without altering the fact that the line $z = \omega'$ intersects pentahedra with peaks at $p_0/q_0, \dots, p_n/q_n$ (a finite number). The bases lying on the sphere $z\bar{z} + \zeta^2 = 1$ cluster in infinite number about each point of the equator. If we put $\omega' = \omega$, where ω is within the limits of variation allowable and where the line $z = \omega$ intersects one of these bases, the suite for ω has an exceptional fraction for which $|p_{m+1}| < |p_m|$. As the pentahedra with peaks at p_{m+1}/q_{m+1} lie within the sphere, this fraction is subsequent to p_n/q_n , whose pentahedra lie without, and the theorem is established.*

(c) If M passes through the origin, M' is a plane. The sphere $S(Q/P)$ is then the reflection of $S(q/p)$ in this plane, and the two spheres are equal. Whence $1/2P\bar{P} = 1/2p\bar{p}$, or

$$(41) \quad |P| = |p|.$$

There are infinitely many spheres of the system passing through the origin. They are the spheres which result from inverting the planes of the system $\xi = m + \frac{1}{2}, \eta = n + \frac{1}{2}, (m, n = 0 \pm 1, \pm 2, \dots)$, in the sphere $z\bar{z} + \zeta^2 = 1$. The line $z = \omega$ can be chosen to intersect as many of these spheres as is desired, but we shall see that the number that can be intersected in bases is limited.

* This result is in striking contrast with that in the case of reals, where such a fraction can appear only as the second fraction of a hermitian suite—thus $1/0, 0/1, \dots$.

It will be easier to consider the planes into which these spheres are transformed. By the transformation $z' = 1/z$ the half-line $z = \omega$ is carried into a semicircle through the origin and the point $1/\omega$. By taking $1/\omega$ sufficiently large the semicircle can be made to intersect as many of the planes as is desired. But the portions of these planes outside the unit spheres of the system are lateral faces of pentahedra with peaks at ∞ , hence the bases lying in these planes are all within these unit spheres. Each of the unit spheres, which has its center always at a complex integer, intersects four planes of the system.

Now if $1/\omega$ lies within one of the four unit spheres passing through the origin the entire semicircle lies within that sphere and can intersect only the four planes intersected by that sphere; and, since it can intersect each plane but once, the number of plane bases traversed is not more than four. If on the other hand $1/\omega$ lies outside one of the four unit spheres through the origin the semicircle enters the fundamental pentahedron by its base before intersecting any of the planes. The semicircle then crosses a number of plane lateral faces of pentahedra and eventually passes out of the pentahedra with peaks at ∞ by the base of one of them. It can then intersect only the planes lying in the sphere forming that base, which are four in number. We have then the rough result that not more than four plane bases can be intersected by the semicircle; hence, the number of terms for which $|P| = |p|$ is not greater than four.*

The following two examples illustrate the three cases just discussed.

Suite for $\omega = \frac{5}{3}$:

$$\frac{1}{0}, \frac{0}{1}, -\frac{1}{4}, -\frac{1}{5}, \frac{2}{9}, \frac{5}{23}.$$

Suite for $\omega = \frac{450 + 30i}{451}$:

$$\frac{1}{0}, \frac{1}{1}, \frac{-1 + 15i}{15i}, \frac{-15i}{-1 - 15i}, \frac{1 - 30i}{-1 - 30i}, \frac{450 + 30i}{451}.$$

Value of the fractions as approximations to ω . It is easy to show that the distance of the approximating fraction from the limit ω tends steadily to zero; that is, that each fraction of the suite is nearer ω than the preceding fraction. For we have noted the fact that P/Q and p/q are inverse points with respect to the circle in which the sphere M cuts the z -plane. The point P/Q lies inside the circle, and is then nearer to any interior point of the circle than is p/q . Since ω is within the circle, P/Q is nearer ω than is p/q .

This fact follows also from the following important theorem due to Hermite:

* It is easy to show that this number is not greater than three, but we omit the proof of this. We can show that one of these fractions can be as far from the beginning of the suite as desired, but that any others must remain near the beginning.

THEOREM. *If p/q is a fraction in the suite for ω and r/s is any other fraction for which $|s| \leq |q|$, then*

$$(42) \quad \left| \omega - \frac{p}{q} \right| < \left| \omega - \frac{r}{s} \right| \cdot \left| \frac{s}{q} \right| \leq \left| \omega - \frac{r}{s} \right|.$$

This theorem is susceptible of a geometric proof, but it lacks the simplicity of the following proof given in Hermite's memoir: Since $x = p, y = q$ is the minimum of the form (1) of the Introduction for some value k_1 of k , we have

$$(p - \omega q)(\bar{p} - \bar{\omega} \bar{q}) + k_1^2 q \bar{q} < (r - \omega s)(\bar{r} - \bar{\omega} \bar{s}) + k_1^2 s \bar{s}.$$

This inequality holds for all integral values of r and s different from p and q . If now $|s| \leq |q|$, $k_1^2 q \bar{q} \geq k_1^2 s \bar{s}$, and therefore

$$(p - \omega q)(\bar{p} - \bar{\omega} \bar{q}) < (r - \omega s)(\bar{r} - \bar{\omega} \bar{s})$$

which yields at once the inequality (42).

There is thus no better rational approximation to ω than a given fraction of Hermite without having a larger denominator. Since $|q| < |Q|$ this shows that P/Q is a better approximation than p/q .

The fraction preceding p/q . Let us consider all numbers whose suite of Hermite contains the fraction p/q , and let us fix our attention upon the fraction immediately preceding p/q in each suite. How many different fractions appear? The discussion of this problem sheds an interesting light upon the geometry of the pentahedral division of the half-space. Since the line $z = \omega$ enters a pentahedron with peak at p/q from a preceding pentahedron by the common base, the number of different fractions is equal to the number of the bases of pentahedra with peaks at p/q which are turned upward. It is also easily shown to be the number of bases whose points of tangency with the sphere $S(p/q)$ lie above the horizontal plane, $\xi = 1/2q\bar{q}$, through the center of that sphere.

Let p_0, q_0 be integers satisfying the equation $pq_0 - p_0q = -1$. Then the transformation of the Group of Picard

$$(43) \quad z' = \frac{pz - p_0}{qz - q_0}$$

transforms the pentahedra with peaks at ∞ into those with peaks at p/q . The pentahedra with bases in common with the preceding have their peaks at the complex integers. Hence, on transforming, the pentahedra with bases in common with those with peaks at p/q have their peaks at $(ps - p_0)/(qs - q_0)$, where s is given all complex integral values. One of these fractions can precede p/q if and only if it has a smaller denominator; that is, if

$$|qs - q_0| < |q|, \quad \text{or} \quad |s - q_0/q| < 1.$$

This leads to the following

THEOREM. Let p_0, q_0 be integers such that $pq_0 - p_0q = -1$. Then the number of different fractions that can immediately precede p/q in a suite of Hermite is equal to the number of complex integers enclosed by a unit circle in the z -plane whose center is q_0/q .

If s_1, s_2, \dots are the values of the complex integers within this circle, then the fractions which can immediately precede p/q are

$$\frac{ps_1 - p_0}{qs_1 - q_0}, \quad \frac{ps_2 - p_0}{qs_2 - q_0}, \quad \dots$$

In Figure 3 a unit square with vertices at complex integers is divided into regions by unit circles with centers at the vertices. A number has been affixed to each region such that if q_0/q lies within the region the number indicates the number of complex integers lying in the unit circle with q_0/q as center. If q_0/q lies at a vertex of the square there is but one integer within the circle (in this case, since q_0/q is an integer, $|q| = 1$, and p/q is an integer), otherwise there are two, three, or four.

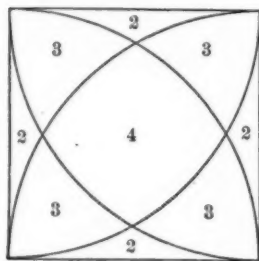


FIG. 3

If p and q are real, p_0 and q_0 can be chosen to be real, and q_0/q lies upon the side of the square of the figure. We observe that in this case the number of integers enclosed is in general two, there

being thus two pentahedra with bases turned upward. Now the plane $\eta = 0$ cuts the pentahedra with peaks on the real axis in a configuration which is precisely the classic modular division of the half-plane. We can then make the following statement:

In the modular division of the half-plane two and only two triangles with their peaks at a point p/q have their bases turned upward, unless p/q is an integer, when there is but one.

Pentahedra with plane bases. Of many other geometrical considerations amenable to the methods of the preceding paragraphs* we shall treat one—to determine the number of bases of the pentahedra with peaks at p/q which are plane. The condition that the base common to pentahedra with peaks at p/q and p'/q' be plane is that $S(p/q)$ and $S(p'/q')$ be equal in magnitude, for these spheres are inverses with respect to the common base. Since q' is of the form $qs - q_0$, this requires that $|qs - q_0| = |q|$, or $|s - q_0/q| = 1$. That is, s is a complex integer on the circumference of the unit circle whose center is q_0/q .

* We can show, for example, that the number of bases whose points of tangency with $S(p/q)$ lie above a plane $\xi = h$ is equal to the number of complex integers lying within a circle whose center is q_0/q and whose radius is a simple function of h . Also, the number of vertices above that plane is equal to the number of mid-points of squares such as Fig. 3 within a suitable circle with q_0/q as center.

Referring to Figure 3 we see that if q_0/q is an integer (in which case p/q is an integer) there are four integers on the circumference. There are no points of the figure on three circles. There are points lying on two circles, but we find on solving that these points are irrational, and hence are impossible values for q_0/q . There are, however, infinitely many rational points on one circle. Whence, *of the bases of pentahedra with peaks at p/q , four are plane if p/q is a complex integer, otherwise not more than one can be plane.*

8. Condition that p/q be a fraction in the suite for ω . We propose in this section to find necessary and sufficient conditions that a given fraction should belong to the hermitian suite of a given number without the necessity of calculating the suite. That is, we shall find the condition that the line $z = \omega$ should intersect pentahedra with peaks at p/q . If the line does not intersect the sphere $\Sigma(p/q)$ within which the pentahedra lie, p/q is not a fraction of the suite. If the line intersects $S(p/q)$, which lies entirely within the pentahedra in question, p/q certainly is a fraction of the suite. These facts give the following statement:

A necessary condition that p/q belong to the suite for ω is that

$$\left| \omega - \frac{p}{q} \right| \leq \frac{1}{\sqrt{2}q\bar{q}};$$

a sufficient condition is that

$$\left| \omega - \frac{p}{q} \right| < \frac{1}{2q\bar{q}}.$$

The case in which $z = \omega$ intersects $\Sigma(p/q)$ but does not intersect $S(p/q)$ remains in doubt and requires a test of much greater nicety. If p/q is a fraction of the suite the line $z = \omega$ must pass through one of the bases of the pentahedra which are turned upward. If these bases are projected orthogonally upon the z -plane ω must lie in the resulting figure. Now the boundaries of these bases are circles orthogonal to the z -plane, and they consequently project into straight lines. The upturned bases then project into a *rectilinear polygon*. This polygon can be constructed when we know the projections, that is, the z -coordinates, of the vertices of the bases.

We see from the preceding section that these upturned bases are the transforms of the bases of pentahedra with peaks at ∞ which lie above the complex integers s_1, s_2, \dots which lie within the unit circle with center at q_0/q , where the transformation to be applied is $z' = (pz - p_0)/(qz - q_0)$. Now the z -coordinates of the vertices of the base above the integer s are the z -coordinates of the vertices of a unit square whose center is s and whose sides are parallel to the real and imaginary axes, and the ζ -coordinate is $\frac{1}{2}\sqrt{2}$. Using the second of equations (5) and putting $z = z_i$, $\zeta = \frac{1}{2}\sqrt{2}$, we get on simplifying

$$(44) \quad z'_i = \frac{p}{q} - \frac{2(\bar{z}_i - \bar{q}_0/\bar{q})}{q^2[2(z_i - q_0/q)(\bar{z}_i - \bar{q}_0/\bar{q}) + 1]}.$$

We can now state the following

THEOREM. Let p_0, q_0 be integers such that $p_0 q_0 - p_0 q = -1$, and let s_1, s_2, \dots be the complex integers lying within the unit circle whose center is q_0/q . Let unit squares with sides parallel to the real and imaginary axes and with centers at s_1, s_2, \dots be constructed. As we pass around the perimeter of the connected figure thus formed, let z_1, z_2, \dots, z_n be the z -coordinates of the successive vertices encountered.

From the values z_1, \dots, z_n determine z'_1, \dots, z'_n by means of equation (44), and form a rectilinear polygon in the z -plane by joining $z'_1 z'_2, z'_2 z'_3, \dots, z'_n z'_1$.

Then the necessary and sufficient condition that p/q be a fraction in the suite for ω is that ω lie within or upon the boundary of this polygon.

We see from the inequalities at the beginning of this section that this polygon will lie within the circle $|z - p/q| = 1/\sqrt{2q\bar{q}}$, and that it will enclose the circle $|z - p/q| = 1/2q\bar{q}$.*

The condition that p_0/q_0 and p/q , where $p_0 q_0 - p_0 q = -1$, are successive fractions in the suite for ω , is that the line $z = \omega$ should intersect the base common to the two pentahedra with peaks at p_0/q_0 and p/q . This base projects into a quadrilateral in the z -plane. Furthermore this base is derived from the base of the fundamental pentahedron by means of the transformation $z' = (pz - p_0)/(qz - q_0)$. The z -coordinates of the vertices of the fundamental pentahedron are $(\pm 1 \pm i)/2$. From these four values we get z'_1, z'_2, z'_3, z'_4 from (44). Then the necessary and sufficient condition that p_0/q_0 and p/q be successive fractions in the suite for ω is that ω lie within or on the boundary of the quadrilateral $z'_1 z'_2 z'_3 z'_4$.

• PART II

REGULAR CONTINUED FRACTIONS

9. Preliminary account. The continued fraction

$$b_0 \pm \frac{\epsilon_1}{b_1 \pm \frac{\epsilon_2}{b_2 \pm \frac{\epsilon_3}{b_3 \pm \dots}}}$$

or as commonly written

$$b_0 \pm \frac{\epsilon_1}{b_1 \pm \frac{\epsilon_2}{b_2 \pm \frac{\epsilon_3}{b_3 \pm \dots}}}$$

* If z'_k is the vertex of this polygon which is farthest from p/q , we have, when p/q is a fraction in the suite for ω ,

$$\left| \omega - \frac{p}{q} \right| \leq \left| z'_k - \frac{p}{q} \right|.$$

This is generally a stronger inequality than (32). I find that $|z'_k - p/q|$ can be as small as but never smaller than $2/3q\bar{q}$.

will be called *regular* if $\epsilon_n = \pm 1$ or $\pm i$ and b_n is a complex integer.* It is clear that by multiplying each ϵ_n and the corresponding denominator by a suitable one of the numbers ± 1 or $\pm i$ the continued fraction can be put in the following form:

$$(45) \quad s_0 - \frac{1}{s_1 - \frac{1}{s_2 - \frac{1}{s_3 - \dots}}},$$

where $s_n (= \pm b_n \text{ or } \pm ib_n)$ is a complex integer, and it is this form that we shall constantly employ.†

The value of the fraction when $\frac{1}{s_n - \frac{1}{s_{n+1} - \dots}}$ is set equal to zero is called the *n*th convergent, and will be represented by p_n/q_n . If the fraction terminates with s_n its value is p_{n+1}/q_{n+1} . If it does not terminate it is said to converge if p_n/q_n approaches a limit as n becomes infinite; and the value or sum of the fraction is defined to be that limit.

The first convergent p_1/q_1 is s_0 , the second p_2/q_2 is $(s_0 s_1 - 1)/s_1$. Putting $p_1 = s_0$, $q_1 = 1$, $p_2 = s_0 s_1 - 1$, $q_2 = s_1$, we note that $p_2 q_1 - p_1 q_2 = -1$. Proceeding from these two convergents, all succeeding convergents can be calculated by means of the recurrence formulæ

$$(46) \quad p_{n+1} = p_n s_n - p_{n-1}, \quad q_{n+1} = q_n s_n - q_{n-1},$$

and the relation

$$(47) \quad p_{n+1} q_n - p_n q_{n+1} = -1$$

always holds. We note that the convergents are all rational fractions in their lowest terms.

If we put $p_0 = 1$, $q_0 = 0$, we observe that the formulæ (46) for $n = 1$ give p_2 and q_2 . We shall then adjoin to the convergents defined above the convergent $p_0/q_0 = 1/0$.

The continued fraction of the suite of Hermite. We can set up at once a regular continued fraction whose convergents are the fractions of a suite of Hermite. The first two fractions of the suite we found to be $1/0$ and $a/1$. In deter-

* The continued fractions involving complex integers have been little studied. Only one kind of such fraction has, so far as I know, appeared in the literature. See Hurwitz, *Acta Mathematica*, vol. 11 (1887), pp. 187-200; Auric, *Journal de mathématiques*, 5th ser., vol. 8 (1902), pp. 387-431.

† The fraction with positive signs

$$b_0 + \frac{1}{b_1 + \frac{1}{b_2 + \frac{1}{b_3 + \dots}}}$$

can be put in the form (45) as follows:

$$b_0 - \frac{1}{-b_1 - \frac{1}{b_2 - \frac{1}{-b_3 - \dots}}},$$

that is, by merely changing the signs of b_1, b_3, b_5, \dots .

mining a fraction P/Q in terms of the two preceding, p'/q' and p/q , an integer s was found and the formulæ (30) used:

$$P = ps - p', \quad Q = qs - q'.$$

These formulæ are identical with (46). Hence, if we put $s_0 = a$ and let s_1, s_2, s_3, \dots be the integers found at each step of the calculation of the fractions, the fractions of the suite are identical with the convergents of the fraction (45).

10. A geometrical interpretation. By means of the S -spheres (Section 4) we can give the continued fraction a geometrical interpretation. Equation (47) shows that $S(p_{n+1}/q_{n+1})$ and $S(p_n/q_n)$ are tangent. The convergents of a regular continued fraction, $1/0, s_0/1, p_2/q_2, p_3/q_3, \dots$, determine a suite of S -spheres, $S(1/0), S(s_0/1), S(p_2/q_2), \dots$, of which the first is the plane $\zeta = 1$, and such that each sphere of the suite is tangent to that which precedes it.

Conversely, if $S(1/0), S(s_0/1), S(p_2/q_2), \dots$ is any suite of S -spheres (the first sphere being $\zeta = 1$) such that each is tangent to that which precedes it, then the points of tangency of the spheres, $1/0, s_0/1, p_2/q_2, \dots$, are the convergents of a regular continued fraction.

To prove the converse we note that the tangency of the S -spheres (see the theorem at the end of Section 4) requires that $p_n q_{n-1} - p_{n-1} q_n = \pm 1$, or $\pm i, n = 1, 2, 3, \dots$. Beginning with the second term of the suite we can multiply numerator and denominator by a suitable one of the numbers ± 1 or $\pm i$ so that $p_n q_{n-1} - p_{n-1} q_n = -1$ always.

Now the transformation $z' = (p_n z - p_{n-1}) / (q_n z - q_{n-1})$ transforms $\zeta = 1$ into $S(p_n/q_n)$; and the spheres tangent to $\zeta = 1$, namely $S(s_m/1)$, where m takes on all complex integral values, are carried into the spheres tangent to $S(p_n/q_n)$. One of these, say $S(s_n/1)$, is carried into $S(p_{n+1}/q_{n+1})$, whence

$$\frac{p_{n+1}}{q_{n+1}} = \frac{p_n s_n - p_{n-1}}{q_n s_n - q_{n-1}},$$

and we can take

$$p_{n+1} = p_n s_n - p_{n-1}, \quad q_{n+1} = q_n s_n - q_{n-1},$$

which are precisely the formulæ (46). We can determine s_n in this manner for each value of n ; and these values put into (45) give a regular continued fraction whose convergents are the z -coördinates of the points of tangency of the suite of spheres.*

We note that if $s_n = 0$ the recurrence formulæ give $p_{n+1}/q_{n+1} = p_{n-1}/q_{n-1}$; that is, $S(p_n/q_n)$ is preceded and followed by the same sphere. Also there is an infinite convergent as often as $S(1/0)$ appears in the suite of S -spheres.

* We can in a like manner give an interpretation by means of Σ -spheres, the suite beginning with $\Sigma(1, 0)$, or $\zeta = \frac{1}{2}\sqrt{2}$, and each sphere intersecting that which precedes it orthogonally.

11. Second geometrical interpretation. Generating curves. We shall now introduce an interpretation which will be more useful for most purposes, an interpretation which is an immediate generalization of that given for the fractions of Hermite in Section 3. We shall treat only the case of convergent continued fractions.

THEOREM. *Let a moving point trace a continuous curve from a point above the plane $\zeta = 1$ to the point $z = \omega$ in the z -plane. Let this curve lie entirely in the space above the z -plane except at the point of termination, and let it intersect no base more than a finite number of times. Then the z -coordinates of the successive peaks of the pentahedra through which the point passes are the convergents in order of a regular continued fraction whose sum is ω .*

Conversely any regular continued fraction converging to the value ω can be generated in this way.

Since the beginning of the curve is in a pentahedron with peak at ∞ , the first convergent is $1/0$ as required. Also since the moving point passes from one pentahedron into another with a different peak by passing through the common base, the condition that successive S -spheres be tangent is satisfied. (If the point passes through a vertex it becomes necessary to choose the next pentahedron properly in order that this be true.)

As a first step toward establishing the convergence let us show that if P is any point on the curve, other than the point of termination, and p/q is the z -coordinate of the peak of the pentahedron in which P lies, there are only a finite number of convergents preceding p/q . The portion of the curve preceding P lies above some plane $\zeta = \epsilon > 0$; and it is easily seen that in any finite region above such a plane there are portions of only a finite number of pentahedra. The proof of this is as follows. The sphere $\Sigma(p_n/q_n)$ in which pentahedra with peaks at p_n/q_n lie has the radius $1/\sqrt{2q_n \bar{q}_n}$, and in any finite part of the plane there are only a finite number of fractions for which q_n has a sufficiently small value that this sphere be large enough to intersect $\zeta = \epsilon$. Further, if the sphere is sufficiently large only a finite number of the pentahedra with peaks at p_n/q_n are intersected by the plane. For a transformation of the form $z' = (q'z - p')/(q_n z - p_n)$ carries the pentahedra with peaks at p_n/q_n into those with peaks at ∞ , and carries $\zeta = \epsilon$ into a sphere of radius $1/2\epsilon q_n \bar{q}_n$ tangent to the z -plane at q'/q_n . This latter sphere obviously intersects only a finite number of pentahedra with peaks at ∞ . Hence $\zeta = \epsilon$ intersects only a finite number of pentahedra with peaks at p_n/q_n .

We conclude from the above that there are only a finite number of bases intersected by the curve before reaching the point P , and by hypothesis each is intersected only a finite number of times. As each passage through a base yields a new convergent, it follows that only a finite number precede the convergent p/q mentioned above.

Secondly, we must show that as P approaches the termination of the curve the corresponding convergent approaches ω . It has just been pointed out that the number of fractions in a finite region—in a given neighborhood of ω for example—whose Σ -spheres have a radius exceeding any small positive quantity, η say, is finite. Unless ω is one of these rationals we can construct a sphere R with ω as center of sufficiently small radius that it intersects none of these Σ -spheres. Then, if p/q is the peak of the pentahedron in which P lies, $\Sigma(p/q)$ has a radius less than η , when P lies within R . As P approaches ω we can decrease the radius of R and likewise let η approach zero. Since $\Sigma(p/q)$ always contains P and since the radius of $\Sigma(p/q)$ approaches zero, the point of tangency, p/q , approaches ω .

If $\omega = p'/q'$, a rational, the above reasoning applies to all Σ -spheres except $\Sigma(p'/q')$, but as the point of tangency of this sphere is equal to ω , we have convergence in this case also.

To show that any convergent regular continued fraction can be defined in this way, we consider the suite of S -spheres, $S(1/0)$, $S(s_0/1)$, $S(p_2/q_2)$, ..., determined by the fraction. Let A be a point above the plane $\xi = 1$, and join A to the point of tangency of $S(1/0)$ and $S(s_0/1)$ by a straight line, then join this point to the point of tangency of $S(s_0/1)$ and $S(p_2/q_2)$ by a straight line. Continuing this process we get a continuous curve composed of segments of straight lines. It is obvious that the convergents determined by this curve are precisely those of the continued fraction.

It remains to show that a point moving along the curve thus determined approaches ω as its terminating value. If the fraction terminates we shall join the last point of tangency to the point ω , the line thus lying entirely within the last S -sphere, and the condition demanded is obviously satisfied.

If the fraction does not terminate its convergence requires that the denominator of the n th convergent p_n/q_n should approach infinity as n increases (excepting the case where $\omega = p'/q'$ and p'/q' appears infinitely often as a convergent). Then the radius of the sphere $S(p_n/q_n)$ approaches zero and a point on the straight line segment within the sphere approaches ω . (If p'/q' appears infinitely often this reasoning applies to all spheres except $S(p'/q')$. But p'/q' can not appear twice in succession as a convergent and the segment within $S(p'/q')$ joins points of tangency of decreasing spheres and this part of the curve also approaches p'/q' or ω .) The theorem is thus established.

12. Connection of the preceding with quadratic forms. Let the curve of the preceding section be represented in parametric form

$$(48) \quad z = \phi_1(t) + i\phi_2(t), \quad \xi = f(t),$$

the interval of the parameter being $t_1 \leq t \leq t_2$. We have then $f(t_1) > 1$, $\phi_1(t_2) + i\phi_2(t_2) = \omega$, $f(t_2) = 0$. By referring to Section 2, we see that the

point on the curve whose parameter is t is the representative point of the definite hermitian form

$$(49) \quad \{x - [\phi_1(t) + i\phi_2(t)]y\} \{\bar{x} - [\phi_1(t) - i\phi_2(t)]\bar{y}\} + [f(t)]^2 y\bar{y}.$$

The proposition of the preceding section can then be put in the following form:

THEOREM. *Let $(p_0, q_0), (p_1, q_1), \dots$ be the successive complex integral values of (x, y) yielding the minimum of the form (49) as t decreases from t_1 to t_2 . Then $p_0/q_0, p_1/q_1, \dots$ are the successive convergents of a continued fraction whose sum is ω .*

The suite of Hermite is an example of this, in which $\phi_1(t) + i\phi_2(t) = \omega$, a constant, and $f(t) = t$.

13. The convergent as an approximation to the sum. From a knowledge of the form of the generating curve we can usually determine the degree of approximation to the sum of the continued fraction which the convergents afford. For example

If when $t < t'$ the generating curve lies always within the right circular cone whose vertex is ω and whose generators make an angle α with the z -plane, then there exists an integer m such that

$$(50) \quad \left| \omega - \frac{p_n}{q_n} \right| < \frac{k}{q_n q_n}, \quad \text{when } n > m,$$

where $k = \frac{1}{2} \sqrt{2} \cot \frac{1}{2} \alpha$.

If when $t < t'$ the generating curve lies always within a sphere of radius a tangent to the z -plane at the point ω , then there exists an integer m such that

$$(51) \quad \left| \omega - \frac{p_n}{q_n} \right| < \frac{k}{|q_n|}, \quad \text{when } n > m,$$

where $k = \sqrt[4]{8} \sqrt{a}$.

In the first, since the curve passes through $\Sigma(p_n/q_n)$ that sphere must intersect the cone in question. The inequality is a direct result of this requirement. It evidently holds after the convergent p_m/q_m corresponding to the point whose parameter is t' . The inequality (51) expresses the fact that $\Sigma(p_n/q_n)$ intersects the sphere of the theorem.

The first theorem is applicable to any generating curve possessing at the point ω a tangent not lying in the z -plane, the second to those curves having simple contact with the z -plane at that point. The inequality satisfied by the fractions of Hermite is deduced at once from (50) in putting $\alpha = \pi/2$. If $\omega = p/q$, a rational fraction, the fraction always terminates in the first case, and terminates in the second case if $a \leq 1/2q\bar{q}$, for then the generating curve after the point t' lies always within pentahedra with peaks at p/q .

THEOREM. *If the generating curve possesses contact of order λ with the z -plane*

at the point ω , there exists a constant k such that all convergents satisfy the inequality

$$(52) \quad \left| \omega - \frac{p_n}{q_n} \right| < \frac{k}{|q_n|^{2/(\lambda+1)}}.$$

Representing for brevity the point on the curve by $z(t)$, $\zeta(t)$, let $a(t)$ be defined by the equation

$$\zeta(t) = a(t) |z(t) - \omega|^\mu.$$

Then $\lim_{t=t_2} a(t) = 0$, or $a \neq 0$, or ∞ , according as μ is less than, or equal to, or greater than $\lambda + 1$. From this it follows that if $a_1 < a$, after a certain point the generating curve lies entirely within the surface of revolution

$$\zeta = a_1 |z - \omega|^{\lambda+1}.$$

After a certain convergent $\Sigma(p_n/q_n)$ must intersect this surface. The surface is tangent to the z -plane at ω , and as convergents are taken closer and closer to ω the surface above them approaches more nearly a plane in form. If we replace a_1 by a still smaller value a_2 , the ratio of the ζ -coordinates of the two surfaces being $\zeta_2/\zeta_1 = a_2/a_1 < 1$, we see that after a certain point in the series of convergents the highest point of the sphere $\Sigma(p_n/q_n)$, viz., $z = p_n/q_n$, $\zeta = \sqrt{2}/q_n \bar{q}_n$, lies always within the latter surface. From this

$$\frac{\sqrt{2}}{q_n \bar{q}_n} > a_2 \left| \frac{p_n}{q_n} - \omega \right|^{\lambda+1}.$$

This is an inequality of the form (52), determining a value of k . We can increase k sufficiently to make the inequality hold for the finite number of convergents at the beginning of the series for which the inequality just found does not hold, and the theorem is established.

It is easy to show that if the curve possesses contact of order λ the convergents do not all satisfy an inequality of the form

$$\left| \omega - \frac{p_n}{q_n} \right| < \frac{k}{|q_n|^\mu}, \quad \text{where} \quad \mu > \frac{2}{\lambda+1}$$

however great k may be.*

14. Formulæ for the calculation of the continued fraction. The determination of the convergents defined by a given generating curve will be a very complicated procedure. This we found to be the case (Section 5) when the

* By the use of a generating curve whose contact exceeds any finite order, such as

$$z(t) = \omega + t, \quad \zeta(t) = e^{-1/t^2},$$

we can establish the existence of convergent continued fractions whose convergents satisfy no inequality of this form, however large k and however small μ (> 0) be chosen.

generating curve is a straight line. We propose to give formulæ for a continued fraction analogous to the "modified hermitian suite" of Section 6. For this purpose we take as the convergent following p/q the z -coördinate of the peak of the pentahedron in which lies the last intersection of the curve with $\Sigma(p/q)$. Since the bases of the pentahedra with peaks at p/q lie very close to $\Sigma(p/q)$ this process will give in most cases in which simple generating curves are employed the convergent immediately following p/q in the original fraction. In any case we shall thus set up a continued fraction differing from the original by the omission of certain convergents, and only a finite number of successive convergents will be omitted.* We note that in this modified continued fraction each convergent appears but once.

A consideration of the curve of rectilinear segments treated in Section 11 shows that if each convergent appears only once the last intersection of the curve with $\Sigma(p_n/q_n)$ lies within $S(p_{n+1}/q_{n+1})$ and thus gives the correct convergent following p_n/q_n . That is, *any regular continued fraction, no two of whose convergents are equal, is the modified continued fraction of some generating curve.*

Let the generating curve have the equations (48). The first convergent will be $1/0$; the second will be the complex integer nearest to the z -coördinate of the last intersection of the curve with the plane $\Sigma(1/0)$, or $\zeta = \frac{1}{2}\sqrt{2}$. As the parameter t is supposed decreasing as we proceed along the curve, the parameter of this point of intersection is the algebraically smallest root, t' , of $f(t) = \frac{1}{2}\sqrt{2}$. The second convergent is s_0 , the nearest integer to $\phi_1(t') + i\phi_2(t')$. We take then $p_0 = 1, q_0 = 0, p_1 = s_0, q_1 = 1$, whence $p_1 q_0 - p_0 q_1 = -1$.

Now let $p_{n-1}/q_{n-1}, p_n/q_n$ be successive convergents, where

$$p_n q_{n-1} - p_{n-1} q_n = -1.$$

We propose to find p_{n+1}/q_{n+1} . The sphere $\Sigma(p_n/q_n)$ has the equation, from (17),

$$(53) \quad \left(z - \frac{p_n}{q_n}\right) \left(\bar{z} - \frac{\bar{p}_n}{q_n}\right) + \zeta^2 - \frac{\sqrt{2}\zeta}{q_n q_n} = 0.$$

Replacing z and ζ by their parametric values, we find that the parameter t_n of the last intersection of the curve with this sphere is the smallest root of the equation

$$(54) \quad [\phi_1(t)^2 + \phi_2(t)^2 + f(t)^2] q_n q_n - [\phi_1(t) + i\phi_2(t)] \bar{p}_n q_n \\ - [\phi_1(t) - i\phi_2(t)] p_n \bar{q}_n - \sqrt{2}f(t) + p_n \bar{p}_n = 0.$$

* Except in the case of a fraction having p/q ($= \omega$) appearing infinitely often, which may be rendered terminating.

On making the transformation $z' = (q_{n-1}z - p_{n-1})/(q_nz - p_n)$, the sphere $\Sigma(p_n/q_n)$ becomes the plane $\zeta = \frac{1}{2}\sqrt{2}$, and the point whose parameter is t_n becomes a point in that plane whose z -coördinate is (employing the second of equations (5))

$$\frac{[\phi_1(t_n)^2 + \phi_2(t_n)^2 + f(t_n)^2]q_{n-1}q_n - [\phi_1(t_n) + i\phi_2(t_n)]\bar{p}_nq_{n-1} - [\phi_1(t_n) - i\phi_2(t_n)]p_{n-1}\bar{q}_n + p_{n-1}\bar{p}_n}{[\phi_1(t_n)^2 + \phi_2(t_n)^2 + f(t_n)^2]q_n\bar{q}_n - [\phi_1(t_n) + i\phi_2(t_n)]\bar{p}_nq_n - [\phi_1(t_n) - i\phi_2(t_n)]p_n\bar{q}_n + p_n\bar{p}_n}.$$

By performing a division and using the fact that t_n is a root of (54) this can be written in the form

$$(55) \quad \frac{q_{n-1}}{q_n} - \frac{\bar{q}_n[\phi_1(t_n) - i\phi_2(t_n)] - \bar{p}_n}{\sqrt{2}q_nf(t_n)}.$$

The peak of the pentahedron in which this point in the plane $\zeta = \frac{1}{2}\sqrt{2}$ lies is at s_n , the nearest integer to (55). Transforming back by the inverse of the above transformation, $z' = (p_nz - p_{n-1})/(q_nz - q_{n-1})$, we can write

$$(56) \quad p_{n+1} = p_ns_n - p_{n-1}, \quad q_{n+1} = q_ns_n - q_{n-1}, \quad \text{whence} \\ p_{n+1}q_n - p_nq_{n+1} = -1.$$

We have then in brief the following method of procedure.

The first two convergents are $p_0/q_0, p_1/q_1$ where $p_0 = 1, q_0 = 0, p_1 = s_0, q_1 = 1$, in which s_0 is the nearest integer to $\phi_1(t') + i\phi_2(t')$ where t' is the smallest root of $f(t) = \frac{1}{2}\sqrt{2}$ in the interval $t_2 \leq t \leq t_1$.

Beginning with these two convergents, the following convergents are calculated by means of the recurrence formulæ (56) where, after t_n has been found as the smallest root of (54) in the interval, s_n is the complex integer nearest to the quantity (55).

The continued fraction with these convergents is then

$$\omega = s_0 - \frac{1}{s_1 - \frac{1}{s_2 - \dots}}.$$

15. An example. The preceding process will involve considerable labor unless the generating curve is a simple one, but it furnishes a method of setting up continued fractions with desired properties. If we choose a straight line or circle as generating curve there are only two intersections of the curve with $\Sigma(p_n/q_n)$ and the determination of t_n involves only the solution of a quadratic equation.

We shall now examine a case in which the generating curve is a circle tangent to the z -plane,

$$(57) \quad z = \omega + \frac{\sqrt{2}t}{1+t^2}, \quad \zeta = \frac{\sqrt{2}t^2}{1+t^2}, \quad 0 \leq t \leq 2.$$

This is of radius $\frac{1}{2}\sqrt{2}$ and lies in a plane parallel to $\eta = 0$. The interval for t is such that half of the circle, from its highest point to the point of tangency ω , is considered.

We find that s_0 is the complex integer nearest to $\omega + \frac{1}{2}\sqrt{2}$. The equation (54) for the determination of t_n is

$$(v_n \bar{v}_n + 2q_n \bar{q}_n - 2)t^2 + \sqrt{2}(v_n \bar{q}_n + \bar{v}_n q_n) + v_n \bar{v}_n = 0,$$

in which $v_n = q_n \omega - p_n$. The roots of this are

$$\frac{-\sqrt{2}(v_n \bar{q}_n + \bar{v}_n q_n) \pm \sqrt{2(v_n \bar{q}_n + \bar{v}_n q_n)^2 - 4v_n \bar{v}_n(v_n \bar{v}_n + 2q_n \bar{q}_n - 2)}}{2(v_n \bar{v}_n + 2q_n \bar{q}_n - 2)}.$$

The denominator of this expression is easily seen to be positive, hence the smallest root is that with the negative sign of the radical.

Putting this value of t_n into (55) and simplifying we find that s_n is the nearest complex integer to the quantity

$$(58) \quad \frac{q_{n-1} - u_n}{q_n} - \frac{u_n}{\bar{u}_n} [iB_n - \sqrt{u_n \bar{u}_n - \frac{1}{2} - B_n^2}],$$

where $u_n = 1/(q_n \omega - p_n)$, and iB_n is the imaginary part of $q_n u_n$.

If ω is real p_n, q_n , and u_n will be real and (58) takes the particularly simple form

$$\frac{q_{n-1} - u_n}{q_n} + \sqrt{u_n^2 - \frac{1}{2}}.$$

Following are two examples of this development:

$$\frac{1}{2} = 1 - \frac{1}{4} - \frac{1}{1} - \frac{1}{4} - \frac{1}{1} - \frac{1}{4} - \dots,$$

$$\frac{1+i}{3} = 1 - \frac{1}{1+2i} - \frac{1}{-1-i} - \frac{1}{-2} - \frac{1}{-1+i} - \frac{1}{1-2i} - \frac{1}{1} - \dots.$$

The radius of the generating circle, $\frac{1}{2}\sqrt{2}$, is sufficiently great so that if $\omega = p/q$, a rational, the generating circle is not contained within $\Sigma(p/q)$; hence no pentahedra with peaks at p/q are intersected. This development, therefore, furnishes always non-terminating continued fractions.

An application of the inequality (51) shows that the convergents always satisfy the inequality

$$\left| \omega - \frac{p_n}{q_n} \right| \leq \frac{\sqrt{2}}{|q_n|}.$$

This can be derived also from (58), for, since the expression under the radical must be positive, $u_n \bar{u}_n \geq \frac{1}{2}$, which is the inequality just found.

The periodicity of these continued fractions will be discussed in a later section.

16. Periodic regular continued fractions. The continued fraction (45) is called *periodic* if integers m and k exist such that $s_{n+k} = s_n$ whenever $n > m$. We shall write the periodic fraction in the form

$$(59) \quad s_0 - \frac{1}{s_1 -} \cdots \frac{1}{s_m -} \frac{1}{s_{m+1} -} \cdots \frac{1}{s_{m+k} -},$$

where the portion between the stars is repeated again and again.

Let us suppose that the fraction converges and that ω is its sum. Let ω_0 represent the convergent fraction

$$(60) \quad \omega_0 = s_{m+1} - \frac{1}{s_{m+2} -} \cdots \frac{1}{s_{m+k} -}.$$

Then

$$(61) \quad \omega = s_0 - \frac{1}{s_1 -} \cdots \frac{1}{s_m -} \frac{1}{\omega_0} = \frac{p_{m+1}\omega_0 - p_m}{q_{m+1}\omega_0 - q_m} = U(\omega_0),$$

where p_m/q_m , p_{m+1}/q_{m+1} are convergents as previously defined of (59). Also

$$(62) \quad \omega_0 = s_{m+1} - \frac{1}{s_{m+2} -} \cdots \frac{1}{s_{m+k} -} \frac{1}{\omega_0} = \frac{p'_k\omega_0 - p'_{k-1}}{q'_k\omega_0 - q'_{k-1}} = V(\omega_0),$$

where p'_{k-1}/q'_{k-1} , p'_k/q'_k are convergents of (60).

Then, employing the usual notation,

$$(63) \quad \omega = U(\omega_0) = UV(\omega_0) = UVU^{-1}(\omega).$$

That is, the transformation $z' = UVU^{-1}(z)$, which is of the Group of Picard since U and V are, carries ω into itself; ω is a fixed point of the transformation. If we write this transformation $z' = (\alpha z + \beta)/(\gamma z + \delta)$, equation (63) exhibits the well-known result that ω is the root of a quadratic equation

$$(64) \quad \gamma z^2 + (\delta - \alpha)z - \beta = 0,$$

whose coefficients are complex integers.

We shall now establish the following important

THEOREM. The transformation $z' = UVU^{-1}(z)$ transforms p_n/q_n into p_{n+k}/q_{n+k} , when $n > m$.

Conversely if integers m and k exist such that

$$\frac{p_{n+k}}{q_{n+k}} = T\left(\frac{p_n}{q_n}\right), \quad \text{when } n > m,$$

where $z' = T(z)$ is a transformation of the Group of Picard, the continued fraction is periodic.

Now p_n/q_n is the value of the fraction when it terminates with s_{n-1} . It is

equal to the second member of (61) if ω_0 be replaced by

$$s_{m+1} - \frac{1}{s_{m+2} - \cdots \frac{1}{s_{n-1}}} \quad (= A, \text{ say}).$$

That is, $p_n/q_n = U(A)$. Similarly

$$\frac{p_{n+k}}{q_{n+k}} = U\left(s_{m+1} - \cdots \frac{1}{s_{m+k} - \frac{1}{s_{m+k+1} - \cdots \frac{1}{s_{n+k-1}}}}\right) = UV(A).$$

Hence, $p_{n+k}/q_{n+k} = UVU^{-1}(p_n/q_n)$. This reasoning does not require the convergence of the continued fraction.

To establish the second part of the theorem, let

$$T(z) = (\alpha z + \beta)/(\gamma z + \delta),$$

where the integers $\alpha, \beta, \gamma, \delta$ satisfy the equation $\alpha\delta - \beta\gamma = 1$. Then, if $n > m$, we have

$$\frac{p_{n+k}}{q_{n+k}} = \frac{\alpha p_n + \beta q_n}{\gamma p_n + \delta q_n}.$$

It is easily shown that both fractions are in their lowest terms. Hence

$$p_{n+k} = \epsilon_n(\alpha p_n + \beta q_n), \quad q_{n+k} = \epsilon_n(\gamma p_n + \delta q_n), \quad \epsilon_n = \pm 1, \pm i.$$

From this and the similar equations for p_{n+k-1}, q_{n+k-1} , we have

$$p_{n+k} q_{n+k-1} - p_{n+k-1} q_{n+k} = \epsilon_n \epsilon_{n-1} (p_n q_{n-1} - p_{n-1} q_n).$$

Since the first member, and also the bracketed part of the second member are each -1 , we have $\epsilon_n \epsilon_{n-1} = 1$. Likewise $\epsilon_{n+1} \epsilon_n = 1$, etc., whence $\epsilon_{m+1}, \epsilon_{m+2}, \dots$ are either all real or all imaginary.

Finally,

$$\begin{aligned} p_{n+k+1} &= \epsilon_{n+1}(\alpha p_{n+1} + \beta q_{n+1}) = \epsilon_{n+1}[\alpha(p_n s_n - p_{n-1}) + \beta(q_n s_n - q_{n-1})] \\ &= \epsilon_{n+1}[(\alpha p_n + \beta q_n)s_n - (\alpha p_{n-1} + \beta q_{n-1})] = \epsilon_{n+1}\left[\frac{p_{n+k} s_n}{\epsilon_n} - \frac{p_{n+k-1}}{\epsilon_{n-1}}\right] \\ &= \epsilon_{n+1} \epsilon_n \left[\frac{p_{n+k} s_n}{\epsilon_n^2} - \frac{p_{n+k-1}}{\epsilon_n \epsilon_{n-1}}\right] = p_{n+k} \frac{s_n}{\epsilon_n^2} - p_{n+k-1}. \end{aligned}$$

Similarly

$$q_{n+k+1} = q_{n+k} \frac{s_n}{\epsilon_n^2} - q_{n+k-1}.$$

Hence $s_{n+k} = s_n/\epsilon_n^2$. If $\epsilon_n = \pm 1$ always, then $s_{n+k} = s_n$; if $\epsilon_n = \pm i$ always, then $s_{n+k} = -s_n$. In the latter case we have however $s_{n+2k} = -s_{n+k} = s_n$. We thus have periodicity in all cases.

17. The character of the transformation T and the resulting properties of the continued fraction. A more intimate knowledge of the periodic fraction

necessitates a few remarks on the different types of linear transformations. Consider a transformation $z' = (\alpha z + \beta)/(\gamma z + \delta)$ where $\alpha\delta - \beta\gamma = 1$. If $\alpha + \delta$ is not real the transformation is called *loxodromic*. The transformations of which $\alpha + \delta$ is real are of three kinds: *elliptic* if $|\alpha + \delta| < 2$, *hyperbolic* if $|\alpha + \delta| > 2$, and *parabolic* if $|\alpha + \delta| = 2$.

The fixed points of the transformation are the roots of equation (64). There is but one fixed point in the case of the parabolic transformation, in all others there are two. We shall treat first those transformations with two fixed points, τ_1 and τ_2 . Let C be the semicircle orthogonal to the z -plane through the two fixed points and lying in the upper half-space.

If the transformation is elliptic it is equivalent to successive inversions in two spheres passing through C .^{*} If we call the dihedral angle between these spheres $\theta/2$ it is necessary that θ be a submultiple of 2π in order that the group be discontinuous. The transform of any sphere through C is a sphere through C making an angle θ with the original. After a finite number of applications of the transformation all points of space are brought back to their original positions.

If T is elliptic there exists an integer ν such that $T^\nu = 1$. Then

$$p_{n+k\nu}/q_{n+k\nu} = p_n/q_n$$

when $n > m$. The convergents p_{m+1}/q_{m+1} , \dots , $p_{m+k\nu}/q_{m+k\nu}$ appear infinitely many times, and the continued fraction does not converge. We shall lay this case aside.

If T is hyperbolic it is equivalent to successive inversions in two spheres with centers in the z -plane and cutting C orthogonally. Each circle C' through τ_1 and τ_2 is transformed into itself. By repetitions of T the transforms of all points of space (one fixed point excepted) approach one of the fixed points ($= \omega$, the sum of the fraction), all the transforms of a point lying on the circle C' through that point.

Let circles C'_1, C'_2, \dots, C'_k be drawn through the fixed points in such a manner that C'_r intersects $\Sigma(p_{m+r}/q_{m+r})$. Now $\Sigma(p_n/q_n)$, $n > m$, is merely the transform by applications of T of one of these k Σ -spheres, hence the k circles intersect all Σ -spheres after $\Sigma(p_m/q_m)$. These circles make angles greater than zero with the z -plane, and we see readily that we can construct a right circular cone with vertex at $z = \omega$ and with axis perpendicular to the z -plane which intersects all the Σ -spheres of the convergents. From this we conclude by the reasoning of the first theorem of Section 13 that there exists a constant k such that all the convergents satisfy the inequality

$$(65) \quad \left| \omega - \frac{p_n}{q_n} \right| < \frac{k}{q_n q_n}.$$

^{*} For the facts mentioned in this section see Poincaré, loc. cit.

If T is loxodromic it is equivalent to the successive performance of two transformations with the fixed points τ_1 and τ_2 , one elliptic, the other hyperbolic, but the angle θ of the former need not be a submultiple of 2π . C is the only fixed circle of the transformation, but we can construct a fixed surface. Let H be the circle of intersection of two spheres orthogonal to C . Then H is a fixed circle for the elliptic component of T . If we form a surface Γ by constructing circles through τ_1 and τ_2 and the points of H this surface is invariant under the transformation T , for the elliptic component of T transforms one of these circles through a point of H into another such circle, and the hyperbolic component leaves the latter circle invariant.

Surfaces $\Gamma_1, \dots, \Gamma_k$ can be constructed to intersect $\Sigma(p_{m+1}/q_{m+1}), \dots, \Sigma(p_{m+k}/q_{m+k})$; and $\Sigma(p_n/q_n)$, $n > m$, will always intersect one of these surfaces. Now H_1, \dots, H_k by means of which these surfaces are constructed will lie entirely in the upper half-space, and the Γ -surfaces will in the neighborhood of ω all lie within a suitable cone with vertex at ω and with axis perpendicular to the z -plane. This cone will intersect all Σ -spheres after a certain point in the suite of convergents, and as before it can be enlarged to intersect all Σ -spheres of the suite. Therefore, a constant k exists such that all the convergents satisfy the inequality (65).

The parabolic transformation presents a very different aspect. In this case there is a single fixed point τ . Any circle through τ and tangent to a certain line in the z -plane is transformed into itself. The transformation is equivalent to successive inversions in two spheres through τ orthogonal to the line just mentioned (and hence tangent to one another).

Let C'_1, \dots, C'_k be members of the family of fixed circles just mentioned such that C'_r intersects $\Sigma(p_{m+r}/q_{m+r})$. Then $\Sigma(p_n/q_n)$, $n > m$, intersects one of these circles. A sphere tangent to the z -plane at $z = \tau = \omega$ and enclosing all these circles can be constructed, and its radius can be enlarged sufficiently to intersect also the Σ -spheres preceding $\Sigma(p_{m+1}/q_{m+1})$. From the reasoning of the second theorem of Section 13 a constant k exists such that the convergents all satisfy the inequality

$$(66) \quad \left| \omega - \frac{p_n}{q_n} \right| < \frac{k}{|q_n|}.$$

It is easily shown that they do not all satisfy an inequality of the form (65) however large k be chosen.

The preceding treatment covers all possible cases. If we add the remark that the fixed points of the hyperbolic and loxodromic transformations are always irrational, whereas the fixed point of the parabolic transformation is always rational, we can state the following

THEOREM. *The convergents of a convergent periodic regular continued fraction*

always satisfy an inequality of the form (66). The sum of the fraction is irrational or rational according as an inequality of the form (65) is or is not satisfied.

18. On quadratic irrationals. It is known from the theory of quadratic forms that there are infinitely many transformations of the Group of Picard leaving invariant the roots of an equation

$$(67) \quad Az^2 + Bz + C = 0,$$

in which A, B, C are complex integers and the discriminant, $D = B^2 - 4AC$, is not a perfect square.* The roots are irrational. If T is one such transformation all powers of T also leave the roots invariant. Since there are only a finite number of elliptic transformations possible, it follows that T can be chosen to be either hyperbolic or loxodromic.

For a better understanding of what follows we shall prove the following

THEOREM. *The necessary and sufficient condition that T can be chosen to be hyperbolic is that $D\bar{D}$ be a perfect square.*

The roots of the given equation being the fixed points of T ,

$$z' = (\alpha z + \beta)/(\gamma z + \delta),$$

they are the roots of the equation

$$\gamma z^2 + (\delta - \alpha)z - \beta = 0.$$

If A, B, C are without a common factor, as we can suppose without loss of generality, we have

$$\gamma = uA, \quad \delta - \alpha = uB, \quad \beta = -uC,$$

where u is a complex integer. If we put $\alpha + \delta = t$, whence

$$\delta = (t + uB)/2, \quad \alpha = (t - uB)/2,$$

the relation $\alpha\delta - \beta\gamma = 1$ yields the well-known equation of Pell

$$t^2 - Du^2 = 4.$$

Conversely any integral solutions of Pell's equation which yield integral values of α and δ provide a transformation leaving the roots of the equation unchanged.

Let T be hyperbolic. Then t is real, and Du^2 is real. Writing $D = m + ni$, $u = a + bi$, and equating to zero the imaginary part of Du^2 , we have

$$(a^2 - b^2)n + 2b^2m = 0.$$

If either m or n is zero, obviously $D\bar{D}$ is a perfect square. Otherwise we can

* Fricke-Klein, *Theorie der automorphen Functionen*, vol. 1, pp. 464-467.

write this in the form

$$(an + bm)^2 = b^2(m^2 + n^2) = b^2 D\bar{D}.$$

We can suppose $b \neq 0$, for $b = 0$ requires $a = 0$, and we have $t = \pm 2$, $u = 0$, yielding the transformation $z' = z$. The equation then shows that $D\bar{D}$ is a perfect square, and the condition is necessary.

To prove it sufficient, we suppose $D\bar{D} = G^2$, where $G (> 0)$ is an integer. The circle in the z -plane with the line joining the roots of (67) as a diameter has its center at the point $-B/2A$ and its radius is $|\sqrt{D}/2A|$, or $\sqrt{G/4A\bar{A}}$. Its equation is

$$\left(z + \frac{B}{2A}\right)\left(\bar{z} + \frac{\bar{B}}{2\bar{A}}\right) - \frac{G}{4A\bar{A}} = 0.$$

The sphere, R , with this circle as equator is the representative sphere of the indefinite hermitian form

$$(2Ax + By)(2\bar{A}\bar{x} + \bar{B}\bar{y}) - Gy\bar{y}.$$

From the theory of these forms it is known that among the portions of this representative sphere lying in the different pentahedra a finite number can be chosen from which all others can be derived by suitable applications of transformations of the Group of Picard.*

Let U be a transformation of the group leaving the roots of (67) unaltered. The circle C , orthogonal to the z -plane through the roots, is a fixed circle of U . Also C lies on the surface of R . Let Π be a pentahedron through which C passes (we use here the single, not the double pentahedron), and let Π_1, Π_2, \dots be the transforms of Π by U, U^2, \dots . Let B, B_1, B_2, \dots be the portions of the surface of R lying in Π, Π_1, Π_2, \dots . Since only a finite number of these are non-congruent (i. e., not derivable from one another by a transformation of the group), we can choose two congruent ones B_m and B_n . The transformation carrying B_m into B_n is that carrying Π_m into Π_n , that is U^{n-m} . If θ is the angle of the elliptic component of U^{n-m} any sphere through C is transformed into another making an angle θ with it. Such a sphere is R , which is transformed into itself; hence $\theta = 0$ or π . If $\theta = 0$, U^{n-m} has no elliptic component and is therefore hyperbolic; if $\theta = \pi$, $U^{2(n-m)}$ is hyperbolic. The transformation T can be chosen to be this hyperbolic transformation, and the condition is thus sufficient.

If $D\bar{D}$ is a perfect square we shall call a root of the equation a *hyperbolic irrational*, otherwise we shall call it a *loxodromic irrational*.

19. Periodicity when ω is a hyperbolic irrational. We propose now to establish certain general theorems concerning periodicity when the continued

* Fricke-Klein, loc. cit., p. 467.

fraction is defined by a generating curve as described in Section 11. We shall treat only curves having a continuously turning tangent in the neighborhood of the point ω . For periodicity, if ω is a quadratic irrational, this tangent cannot lie in the z -plane. Let ω' be the other root of the quadratic equation of the form (67) satisfied by ω , and let T be a hyperbolic transformation with ω, ω' as fixed points and let all points be carried to the neighborhood of ω by repeated applications of T .

Let C' be a circle through ω, ω' and tangent to the generating curve at ω . Then C' is a fixed circle of T , and an arc A of C can be found such that all subsequent parts of the circle in the upper half-space are derived from this arc by applications of T, T^2 , etc. Hence the subsequent pentahedra through which C' passes are derived from those through which A passes by these transformations, and the same is true of their peaks.

Now suppose that A does not intersect an edge adjoining a base and is not tangent to a base. Then a small circle K can be drawn with a point on A as center and lying in a plane perpendicular to the arc at this point such that the surface formed by circles through ω, ω' and the points of K intersects always the same bases and in the same order as A . This surface being also invariant under T intersects throughout its whole length the same bases and in the same order as C' . In the infinitesimal neighborhood of ω this surface is a cone surrounding the tangent to the generating curve. After a certain point the generating curve lies entirely within this surface, and intersects the same pentahedra as C' . The z -coordinates of the peaks then satisfy the relation $p_{n+k}/q_{n+k} = T(p_n/q_n)$ for a suitable value of k , and the fraction is periodic. Hence, the following

THEOREM. *If at the hyperbolic irrational ω the circle through ω, ω' tangent to the generating curve does not intersect an edge bounding a base of some pentahedron and is not tangent to a base the continued fraction is periodic.*

If C' intersects an edge adjoining a base the generating curve may in the neighborhood of ω intersect the transforms of any of the bases adjoining this edge. If however a surface be formed by circles through ω, ω' and the edge this surface will divide the cone at ω into two parts, and if the generating curve lies after a certain point always within one of these compartments it will intersect always the transforms of the same base and the fraction will be periodic. This will always occur if the generating curve is algebraic unless it lies entirely upon the algebraic surface just formed, for it can intersect that surface only a finite number of times. The case in which C' is tangent to a base can be treated in a similar manner.

The preceding theorem requires alteration for the modified continued fraction for which formulæ were given in Section 14. Here the convergent following p/q does not depend upon the intersection of the curve with a base

but with the sphere $\Sigma(p/q)$. The points of this spherical surface within the circular quadrilateral formed by the intersection of the four lateral faces of a pentahedron with peak at p/q all lie within pentahedra having the same peak. If we replace the condition of the theorem by the condition that C' shall "not intersect the circle in which the lateral face of some pentahedron intersects the sphere of its vertices" the theorem holds. As before this condition can in general be dispensed with if the generating curve is algebraic.

20. Periodicity when ω is a loxodromic irrational. If T is loxodromic the only fixed circle is the circle C through ω , ω' orthogonal to the z -plane. An arc A of C can be found such that the part of the circle between A and ω is derived from A by the transformations T , T^2 , etc. Suppose that A intersects no edge bounding a base. [Since C is orthogonal to the z -plane it cannot be tangent to a base.] About a point of A let a small circle H , as in Section 17, be constructed—that is, the circle of intersection of two spheres orthogonal to C . The surface Γ formed by circles through ω , ω' and the points of H is invariant under T . Further, if H be sufficiently small this surface intersects the same bases and in the same order as A , and hence the same bases as C throughout its entire course. Now let H be enlarged until Γ intersects an edge or becomes tangent to a base. Call this surface Γ_0 . In the infinitesimal neighborhood of ω , Γ_0 is a right circular cone with the line $z = \omega$ as axis. We have the

THEOREM. *If ω is a loxodromic irrational and the circle C through ω , ω' orthogonal to the z -plane does not intersect an edge bounding a base, there exists a right circular cone Γ_0 with vertex at ω and axis perpendicular to the z -plane, such that if the tangent to the generating curve lies within Γ_0 the fraction is periodic, and if it lies without the fraction is not periodic.*

To prove the latter part of the theorem let H be increased until the tangent to the generating curve is tangent to Γ at the point ω . Now Γ will intersect certain bases not intersected by C . Let B be one of these bases. The transforms of B by T , T^2 , \dots approach ω and owing to the elliptic component of T are rotated about the circle C through multiples of an angle incommensurable with 2π . Infinitely many of these are intersected by the generating curve. But no transformation U exists such that each one intersected is derived from the preceding by an application of U . For all would then be derived from the first by U , U^2 , \dots , and since the angle of the elliptic component of U must be incommensurable with 2π certain of these bases are on the opposite side of C from the generating curve and are not intersected by it. The fraction cannot then be periodic.

The theorem relative to the modified continued fraction is stated in the same terms as before—by substituting "the circle in which a lateral face of a pentahedron intersects the sphere of its vertices" for "an edge bounding a base." The surface Γ_0 is in general different.

Periodicity of the continued fraction derived from the suite of Hermite. The preceding theorems establish the facts concerning the continued fraction whose convergents are the fractions of Hermite. The generating curve is algebraic and even when C' intersects an edge does not lie on the exceptional surface mentioned in Section 19. Furthermore it is perpendicular to the z -plane and lies always inside the cone Γ_0 of Section 20 when that cone exists. Hence the

THEOREM. *The continued fraction whose convergents are the fractions of Hermite is always periodic when ω is a hyperbolic irrational. It is periodic when ω is a loxodromic irrational except when the circle through ω , ω' orthogonal to the z -plane intersects an edge adjoining a base of some pentahedron.*

21. Periodicity when ω is rational. If $\omega = p/q$, a rational, we know that the fraction terminates unless the tangent to the curve at ω lies in the z -plane, and that it is not periodic if the curve has contact with the z -plane of higher order than the first. We shall treat then curves of simple contact. Let C be the osculating circle of the curve at ω . Then C is tangent to the z -plane, but does not lie in that plane. Obviously if C lies entirely within the pentahedra with peaks at p/q the same will be true of the generating curve after a certain point and the fraction will terminate.

Under what circumstances is C a fixed circle for some parabolic transformation with ω as fixed point? These transformations are those carrying some pentahedron with peak at p/q into some other. If we make the transformation $z' = (q_0 z - p_0)/(qz - p)$, where $p q_0 - p_0 q = -1$, the pentahedra with peaks at p/q become those with peaks at ∞ . The circle C becomes a straight line L parallel to the z -plane. The parabolic transformations carrying the pentahedra with peaks at ∞ into themselves are $z' = z + \beta$ where β is a complex integer. The fixed lines are those parallel to the line joining the origin to the point $z = \beta$; that is, parallel to $\bar{\beta}z - \beta\bar{z} = 0$, or if we like parallel to the line

$$\bar{\beta}(z - q_0/q) - \beta(\bar{z} - \bar{q}_0/\bar{q}) = 0$$

through q_0/q , which is parallel to the preceding. The fixed lines for the transformation $z' = z + \beta$ meet this line at ∞ . Transforming back (putting $(q_0 z - p_0)/(qz - p)$ for z in this equation) we get the line

$$\beta q^2(z - p/q) - \bar{\beta} \bar{q}^2(\bar{z} - \bar{p}/\bar{q}) = 0.$$

The parallel lines previously mentioned become circles through p/q tangent to this line. Giving β all integral values we get the lines to one of which C must be tangent in order to be a fixed circle of some parabolic transformation with p/q as fixed point.

The preceding line is not less general on account of the factor q^2 . Putting

$\beta = \bar{\beta}' \bar{q}^2$, an integer, we have

$$\bar{\beta}'(z - p/q) - \beta'(\bar{z} - \bar{p}/\bar{q}) = 0.$$

That is, the tangents to the fixed circles at p/q are parallel to the lines joining the origin to all complex integers β' . Otherwise put, it is necessary and sufficient that the slope, $d\eta/d\xi$, of the tangent to the circle be rational.

Let the slope $d\eta/d\xi$ of the tangent to the generating curve be rational. Then C is a fixed circle of some transformation T , and we see just as in the preceding cases that the peaks of the pentahedra through which C passes fall into a recurrence scheme $p_{n+k}/q_{n+k} = T(p_n/q_n)$. If C intersects no edge bounding a base and is tangent to no base a circle K can be constructed with a point on C as center and lying in a plane perpendicular to C , such that the circles tangent to C at the point ω and passing through the points of K form a surface which intersects the same bases and in the same order as does C . These circles possess simple contact with C whereas the generating curve has contact of a higher order, hence after a certain point the generating curve lies entirely within this surface. Its convergents then satisfy the relation stated above, and the fraction is thus periodic.

If C intersects an edge or is tangent to a base we cannot infer periodicity. If however the generating curve is algebraic we have, as in the hyperbolic case, periodic fractions in general in this case also.

If the tangent to the generating curve has not a rational slope the fraction is not periodic, for the convergents can not satisfy after a certain point a relation of the form $p_{n+k}/q_{n+k} = T(p_n/q_n)$. This is easiest seen by considering the line L and the transformations $z' = z + \beta$. If the slope of L is not rational the repeated application of a transformation of this form will carry any pentahedron which L intersects to a distance as great as we like from L .

We have established then the following

THEOREM. *If the generating curve has simple contact with the z -plane at the rational point ω , the fraction terminates if the osculating circle C of the curve at the point ω lies within pentahedra with peaks at ω . It is not periodic if the slope, $d\eta/d\xi$, of the tangent to the generating curve is irrational.*

If the slope is rational the fraction is periodic unless C intersects an edge adjoining a base of some pentahedron or is tangent to some base.

A similar theorem holds for the modified continued fraction with the change of phraseology previously mentioned.

The periodicity of the continued fraction of Section 15. Here the osculating circle is the generating curve itself. Its radius, $\frac{1}{2}\sqrt{2}$, is sufficiently great to prevent the termination of the fraction. The circle lies in a plane parallel to $\eta = 0$, hence $d\eta/d\xi = 0$, a rational. The fraction is periodic unless the

circle meets the circle in which the lateral face of some pentahedron intersects the sphere of its vertices. This exception is a trivial one. For the generating circle likewise intersects the successive transforms of this circle on the sphere of vertices. We have thus at regular intervals in the formation of the fraction two possible values of the next convergent at our disposal. If the choice be properly made each time the fraction will be periodic. With this understanding for the exceptional case, *the continued fraction is periodic for all rational values of ω .*

22. The number field $[1, \epsilon]$. All the results of this paper can, with suitable modifications, be extended to the number field in which an integer is defined to be a number of the form $m + n\epsilon$, where m and n are real integers and $\epsilon = \frac{1}{2} + \frac{1}{2}i\sqrt{3}$. Bianchi* has found the fundamental polyhedron of the group $z' = (\alpha z + \beta)/(\gamma z + \delta)$, $\alpha\delta - \beta\gamma = 1$, where $\alpha, \beta, \gamma, \delta$ are integers of this field. Three of the polyhedra into which the half-space is divided form the heptahedron whose lateral faces are the six planes $\xi = \pm \frac{1}{2}, \xi + \eta\sqrt{3} = \pm 1, \xi - \eta\sqrt{3} = \pm 1$, and whose base is the unit sphere $z\bar{z} + \zeta^2 = 1$. This heptahedron plays the rôle of the fundamental double pentahedron of the preceding pages.

THE UNIVERSITY OF EDINBURGH,
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ON THE IMAGINARY ROOTS OF A POLYNOMIAL AND THE REAL ROOTS OF ITS DERIVATIVE*

BY

HENRY BEDINGER MITCHELL

1. **Introduction.** If a polynomial with real coefficients is found to have bend points exterior to the interval within which its real roots must lie, or if its derivative vanishes more than once in an interval between two consecutive real roots, the existence of imaginary roots is immediately evident. It would seem probable that such marked features of the graph, as thus instantly reveal the presence of imaginary roots, should also set limits, in the complex plane, to a region within which one or more of these roots must be contained; and the question arises as to what we can infer, without laborious calculation, as to their position. In the theorems which follow, we shall seek a partial answer to this question, and shall therefore be concerned, not with the problem of obtaining the closest possible approximations, but rather with the inferences that can be drawn from certain minima data, that are to include the existence of a bend point, where the curve is not concave toward the X -axis, and its position relative to the adjacent real root or roots.

2. **Notation.** Let

$$(1) \quad f(x) \equiv A_0 x^n + A_1 x^{n-1} + \cdots + A_{n-1} x + A_n$$

be a polynomial having real coefficients, and let

$$(2) \quad \phi(x) \equiv (x - r_1)(x - r_2) \cdots (x - r_m) \quad (r_1 \leq r_2 \leq \cdots \leq r_m),$$

$$(3) \quad \psi(x) \equiv [(x - a_1)^2 + b_1^2] \cdots [(x - a_k)^2 + b_k^2] \quad (2k = n - m)$$

be the products of the factors corresponding, respectively, to its real and imaginary roots, so that

$$f(x) \equiv A_0 \phi(x) \cdot \psi(x).$$

We shall then denote by $L(x)$ and $\lambda(x)$, respectively, the logarithmic derivatives of $\phi(x)$ and $\psi(x)$, so that

$$(4) \quad \frac{f'(x)}{f(x)} \equiv L(x) + \lambda(x),$$

* Presented to the Society, October 28, 1916.

where

$$(5) \quad L(x) = \sum_{i=1}^m \frac{1}{x - r_i},$$

$$(6) \quad \lambda(x) = \sum_{i=1}^k \frac{2(x - a_i)}{(x - a_i)^2 + b_i^2}.$$

3. **Theorem I.** *If $f(x)$ be a polynomial having real coefficients and at least one real root, and if f' have a real root ρ exterior to the interval $r_1 \leq x \leq r_m$, between the least and the greatest real root of f , then there exists at least one pair of imaginary roots of f , $a \pm ib$, whose real and imaginary parts satisfy the inequalities*

$$(7) \quad \begin{aligned} 0 < a - \rho < 2h(\rho - r_m), \\ 0 < b \leq h(\rho - r_m) \end{aligned} \quad (\text{when } \rho > r_m),$$

or

$$(8) \quad \begin{aligned} 0 < \rho - a < 2h(r_1 - \rho), \\ 0 < b \leq h(r_1 - \rho) \end{aligned} \quad (\text{when } \rho < r_1),$$

where $2h$ is the number of imaginary roots whose real part is greater than ρ in the first case, and less than ρ in the second case.

That f has imaginary roots is an immediate consequence of Rolle's Theorem. As ρ is a root of f' that is not a root of f , we may write, from (4),

$$L(\rho) + \lambda(\rho) = 0.$$

When $\rho > r_m$ this gives

$$0 < \frac{1}{\rho - r_m} \leq \sum_{i=1}^m \frac{1}{\rho - r_i} = 2 \sum_{i=1}^k \frac{a_i - \rho}{(a_i - \rho)^2 + b_i^2}.$$

The separate terms of the last sum are positive or negative according as $a_i \geq \rho$, and as the whole sum is positive at least one of its terms must be so also. Denoting by $h \leq k$ the number of such positive terms, and by $a \pm ib$ the pair of imaginary roots corresponding to a term that is at least as great as any other, we have

$$(9) \quad 0 < \frac{1}{\rho - r_m} \leq 2h \frac{a - \rho}{(a - \rho)^2 + b^2} < \frac{2h}{a - \rho},$$

whence

$$0 < a - \rho < 2h(\rho - r_m).$$

To obtain the inequality for b we have only to note that

$$\frac{2(a - x)}{(a - x)^2 + b^2}$$

has the maximum value of $1/b$, so that

$$2h \frac{a - \rho}{(a - \rho)^2 + b^2} \leq \frac{h}{b},$$

and therefore, by (9),

$$0 < b \leq h(\rho - r_m),$$

which completes the proof for the case where $\rho > r_m$, and a precisely similar procedure establishes the inequalities (8) for $\rho < r_1$.

COROLLARY. As $1 \leq h \leq k$, where $2k$ is the number of imaginary roots of f , we may substitute k for h in (7) and (8), which gives the form in which the theorem must be used when we have no indication of how the imaginary roots of f are separated.

4. Theorem II. If $f(x)$ be a polynomial having real coefficients and at least one real root, and if f' have a real root ρ exterior to the interval $r_1 \leq x \leq r_m$ between the least and the greatest real root of f , then f' has an even number of such roots, without and on the same side of the interval, and there exists at least one pair of imaginary roots of f , $\alpha \pm i\beta$, whose real and imaginary parts satisfy the inequalities

$$(10) \quad \begin{aligned} |\rho_2 - \alpha| &< \beta \leq (\rho_2 - r_m) \sqrt{2k}, \\ |\rho_2 - \alpha| &\leq (\rho_2 - r_m) \frac{\sqrt{k}}{2}, \end{aligned}$$

where $\rho_2 > r_m$ is the greatest real root of f' , or

$$(11) \quad \begin{aligned} |\rho_2 - \alpha| &< \beta \leq (r_1 - \rho_2) \sqrt{2k}, \\ |\rho_2 - \alpha| &\leq (r_1 - \rho_2) \frac{\sqrt{k}}{2}, \end{aligned}$$

where $\rho_2 < r_1$ is the least real root of f' , and $2k$ is the number of imaginary roots of f .

It will be sufficient, as before, to establish the first set of inequalities, the proof for the second case being similar save for the change of sign. From (4) we have

$$\frac{f'}{f} = L + \lambda = \sum_{i=1}^m \frac{1}{x - r_i} + \sum_{i=1}^k \frac{2(x - a_i)}{(x - a_i)^2 + b_i^2}.$$

For values of x greater than both r_m and the greatest a in λ , both L and λ are positive. For sufficiently small values of the positive quantity δ , $L + \lambda$ is also positive when $x = r_m + \delta$ and is finite and continuous for all greater values of x . Therefore if f' , and so $L + \lambda$, have one real root greater than r_m it must have an even number of such roots, and at the greatest of these the slope of $L + \lambda$ cannot be negative.

We have, therefore,

$$L'(\rho_2) + \lambda'(\rho_2) \geq 0,$$

whence

$$0 < \frac{1}{(\rho_2 - r_m)^2} \leq \sum_{i=1}^m \frac{1}{(\rho_2 - r_i)^2} \leq 2 \sum_{i=1}^k \frac{b_i^2 - (\rho_2 - a_i)^2}{[b_i^2 + (\rho_2 - a_i)^2]^2}.$$

As the last sum is positive, at least one of its terms must be greater than zero. Denoting by $\alpha \pm i\beta$ the pair of imaginary roots corresponding to a positive term that is at least as great as any other, we may write

$$\beta^2 > (\rho_2 - \alpha)^2,$$

and

$$0 < \frac{1}{(\rho_2 - r_m)^2} \leq 2k \frac{\beta^2 - (\rho_2 - \alpha)^2}{[\beta^2 + (\rho_2 - \alpha)^2]^2}.$$

If we regard the last expression as a function of β , its maximum value is found to be

$$\frac{k}{4(\rho_2 - \alpha)^2}$$

and we observe that it is also less than, or at most equal to, $2k/\beta^2$, from which we obtain

$$0 < \frac{1}{(\rho_2 - r_m)^2} \leq \frac{k}{4(\rho_2 - \alpha)^2},$$

$$\frac{1}{(\rho_2 - r_m)^2} \leq \frac{2k}{\beta^2},$$

whence

$$|\rho_2 - \alpha| \leq (\rho_2 - r_m) \frac{\sqrt{k}}{2},$$

$$|\rho_2 - \alpha| < \beta \leq (\rho_2 - r_m) \sqrt{2k}.$$

5. Theorem III. If a polynomial, $f(x)$, having real coefficients, have at least one real root and not more than two imaginary roots, and if f' have a real root ρ greater than the greatest real root of f or less than its least real root, then f' has two and only two such roots, ρ_1 and ρ_2 , both being greater than the greatest real root of f or both less than its least real root, and f has a pair of imaginary roots, $a \pm ib$, such that

$$(12) \quad 0 < a - \rho_2 < b \leq \rho_1 - r_m \quad \text{if} \quad r_m < \rho_1 \leq \rho_2,$$

or

$$(13) \quad 0 < \rho_2 - a < b \leq r_1 - \rho_1 \quad \text{if} \quad \rho_2 \leq \rho_1 < r_1,$$

so that, of the projections on the X -axis of the two exterior bend points, the one lies nearer to the real part of the imaginary roots of f than the other does to the nearest real root of f .

It will be sufficient to observe that as f has but a single pair of imaginary roots, the inequalities of Theorems I and II must alike apply to that pair for $h = k = 1$.

COROLLARY. By Theorem II, we may also write for $k = 1$,

$$(14) \quad \rho_2 < a \leq \frac{3\rho_2 - r_m}{2} \quad \text{when} \quad r_m < \rho_1 \leq \rho_2,$$

$$(15) \quad \rho_2 > a \geq \frac{3\rho_2 - r_1}{2} \quad \text{when} \quad \rho_2 \leq \rho_1 < r_1,$$

so that, when f has but a single pair of imaginary roots, if the interval on the X -axis from the most remote exterior real root of f' to the nearest real root of f be divided externally in the ratio of $-1:3$, the real part of the imaginary roots of f will lie between the root of f' and the point of division.

6. **Theorem IV.** If $f(x)$ be a polynomial having real coefficients and not more than eight imaginary roots, and if f' reduce to zero more than once in an interval between two distinct, consecutive, real roots of f , $r_h < x < r_i$, then there exists a pair of imaginary roots of f , $\alpha \pm i\beta$, such that α is included in the interval, and β is not greater than its extent:

$$(16) \quad r_h < \alpha < r_i, \quad \text{and} \quad 0 < \beta \leq r_i - r_h.$$

As f' , and so $L + \lambda$, reduces to zero more than once in the interval, there must be an intermediate point $x = c$, for which $L'(c) + \lambda'(c) = 0$, or

$$(17) \quad \sum_{i=1}^m \frac{1}{(c - r_i)^2} = 2 \sum_{i=1}^k \frac{b_i^2 - (c - a_i)^2}{[b_i^2 + (c - a_i)^2]^2}.$$

But

$$(18) \quad \sum_{i=1}^m \frac{1}{(c - r_i)^2} \geq \frac{1}{(c - r_h)^2} + \frac{1}{(c - r_i)^2} \geq \frac{8}{(r_i - r_h)^2}.$$

As the first member of (17) is positive, one or more of the terms of the second member must be positive also. Denoting by $\alpha \pm i\beta$ the pair of imaginary roots of f corresponding to a positive term that is at least as great as any other, we have, as in Theorem II, remembering that $2k \leq 8$,

$$(19) \quad 2 \sum_{i=1}^k \frac{b_i^2 - (c - a_i)^2}{[b_i^2 + (c - a_i)^2]^2} \leq 2k \frac{\beta^2 - (c - \alpha)^2}{[\beta^2 + (c - \alpha)^2]^2} \leq \frac{k}{4(c - \alpha)^2} \leq \frac{1}{(c - \alpha)^2},$$

$$2k \frac{\beta^2 - (c - \alpha)^2}{[\beta^2 + (c - \alpha)^2]^2} \leq \frac{2k}{\beta^2} \leq \frac{8}{\beta^2}.$$

Combining (19) with (17) and (18), we have

$$\beta \leq r_i - r_h, \quad \frac{1}{(c - \alpha)^2} \geq \frac{1}{(c - r_h)^2} + \frac{1}{(c - r_i)^2}.$$

Thus $1/(c - \alpha)^2$ is greater than both $1/(c - r_h)^2$ and $1/(c - r_i)^2$, and hence $|c - \alpha|$ is less than $|c - r_h|$ and less, also, than $|c - r_i|$. As c lies within the interval from r_h to r_i , so also does α .

COROLLARY. When $f(x)$ has but two imaginary roots, then

$$(20) \quad \beta \leq \frac{r_i - r_h}{2}.$$

7. Theorem V. If $f(x)$ be a polynomial having real coefficients, and if f' vanish more than once in an interval $r_h < x < r_i$ between two distinct, consecutive real roots of f , the real parts of the imaginary roots of f cannot all lie without and on the same side of the interval, unless the number of real roots on the opposite side be greater than eight,* and unless the number of imaginary roots be also greater than eight, the degree of the polynomial being thus at least twenty.

The necessary and sufficient condition that f' should vanish more than once in the interval is that there should be a point c , $r_h < c < r_i$, for which

$$L(c) + \lambda(c) = 0, \quad L'(c) + \lambda'(c) \geq 0.$$

As $L'(x) < 0$ for all values of x , we may rewrite these conditions in the form

$$(21) \quad \lambda(c) = -L(c), \quad \left| \frac{\lambda(c)}{\lambda'(c)} \right| \leq \left| \frac{L(c)}{L'(c)} \right|, \quad \lambda'(c) > 0,$$

from which it follows that there must be a point of intersection of the two curves $y = \lambda(x)$, $y = -L(x)$, at which the former is rising, and where its subtangent is less than or at most equal to the subtangent of the latter in absolute value.

The subtangent to λ at any point is

$$\frac{\lambda}{\lambda'} = \frac{2 \sum_{i=1}^k \frac{x - a_i}{(x - a_i)^2 + b_i^2}}{2 \sum_{i=1}^k \frac{b_i^2 - (x - a_i)^2}{[b_i^2 + (x - a_i)^2]^2}}.$$

As we require that every a is to be without and on the same side of the interval, the terms of the numerator are all of the same sign for $x = c$, and, denoting by α the a nearest to c , we may write

$$(22) \quad \begin{aligned} |\lambda(c)| &\geq 2 \sum_{i=1}^k \frac{|c - \alpha|}{(c - a_i)^2 + b_i^2} = 2|c - \alpha| \sum_{i=1}^k \frac{1}{(c - a_i)^2 + b_i^2}, \\ |\lambda'(c)| &< 2 \sum_{i=1}^k \frac{b_i^2 + (c - a_i)^2}{[b_i^2 + (c - a_i)^2]^2} = 2 \sum_{i=1}^k \frac{1}{(c - a_i)^2 + b_i^2}, \\ \left| \frac{\lambda(c)}{\lambda'(c)} \right| &> |c - \alpha|. \end{aligned}$$

* The opposite end-point of the interval, r_h or r_i , is included in this enumeration.

Thus the subtangent to λ at the point c is greater in absolute value than the distance between c and the nearest real part of an imaginary root of f , and as all of these are to lie without the interval while c lies within it, we must have

$$\frac{-\lambda(c)}{\lambda'(c)} > \alpha - c > r_i - c, \text{ if } c < \alpha \leq a_1, a_2, \dots, a_k, \text{ so that } \lambda(c) < 0,$$

$$\frac{\lambda(c)}{\lambda'(c)} > c - \alpha > c - r_h, \text{ if } c > \alpha \geq a_1, a_2, \dots, a_k, \text{ so that } \lambda(c) > 0.$$

But by (21) this requires that we should also have

$$(23) \quad \frac{-L(c)}{L'(c)} > r_i - c, \text{ if } L(c) > 0,$$

$$(24) \quad \frac{L(c)}{L'(c)} > c - r_h, \text{ if } L(c) < 0.$$

We proceed, therefore, to determine the conditions under which these latter inequalities can subsist; i. e., in which the tangent drawn to $y = L(x)$ at an interior point of the interval can intersect the X -axis outside of that interval.

The intercept on the X -axis of the tangent to $y = L(x)$,

$$(25) \quad \begin{aligned} X &= x - \frac{L}{L'} \\ &= x + \frac{\frac{1}{x-r_1} + \dots + \frac{1}{x-r_h} + \frac{1}{x-r_i} + \dots + \frac{1}{x-r_m}}{\frac{1}{(x-r_1)^2} + \dots + \frac{1}{(x-r_h)^2} + \frac{1}{(x-r_i)^2} + \dots + \frac{1}{(x-r_m)^2}}, \end{aligned}$$

may, for a fixed value of x , be regarded as a function of the roots, r_1, \dots, r_m . Taking the partial derivative of X with respect to any one of them, we have

$$(26) \quad L'^2 \cdot \frac{\partial X}{\partial r} = -\frac{L'}{(x-r)^2} - \frac{2L}{(x-r)^3}.$$

As $L' < 0$ for all values of x and r , the first term of the second member is always positive. When $L > 0$ and $r > x$, or when $L < 0$ and $r < x$, the second term is also positive and $(\partial X / \partial r) > 0$. Therefore for any point where L is positive, X will be increased by removing indefinitely to the right any real root of f that lies to the right of that point; and for any value of x for which L is negative, X will be made less if we remove indefinitely to the left any real root that lies to the left of x . But to remove a root indefinitely to the right or left is ultimately to cause the corresponding term in L and L' to vanish. Therefore we obtain an upper and a lower limit for the intercept

of the tangent drawn to $y = L(x)$ at any point of the interval by disregarding, for the upper limit, all the real roots greater than r_i , and, for the lower limit, all the real roots less than r_h .

But when $L > 0$ and $r < x$, or when $L < 0$ and $r > x$, the second member is composed of two terms of opposite sign, and becomes zero when

$$r = x + \frac{2L}{L'},$$

which gives us the maximum value of X in the first case, and the minimum in the second. Substituting this value for each r that is less than x in the expression (25) for X , and removing indefinitely to the right all the real roots greater than r_i , we have, as the greatest value which X can assume when L is positive,

$$X = x - \frac{L}{L'} = x + \frac{-h \frac{L'}{2L} + \frac{1}{x - r_i}}{h \frac{L'^2}{4L^2} + \frac{1}{(x - r_i)^2}},$$

where h denotes the number of real roots of f that lie to the left of x , including the root r_h at the beginning of the interval. This gives us,

$$-\frac{L}{L'} \left(h \frac{L'^2}{4L^2} + \frac{1}{(x - r_i)^2} \right) = -h \frac{L'}{2L} + \frac{1}{x - r_i},$$

$$\frac{h}{4} \frac{L'}{L} - \frac{1}{x - r_i} - \frac{L}{L'} \frac{1}{(x - r_i)^2} = 0,$$

or, for $x = c$,

$$(27) \quad \frac{h}{4} = \frac{1}{r_i - c} \left(-\frac{L(c)}{L'(c)} \right) + \frac{1}{(r_i - c)^2} \left(\frac{L(c)}{L'(c)} \right)^2.$$

If now, as (23) requires, we are to have

$$-\frac{L(c)}{L'(c)} > r_i - c,$$

each of the terms of the second member of (27) must be greater than unity, and therefore

$$(28) \quad \frac{1}{4} h > 2, \quad h > 8,$$

and the degree of ϕ must be at least 10.

A precisely similar substitution shows that X cannot fall to the left of r_h unless the number of real roots to the right of the interval is greater than 8, and so, as before, the total number of real roots is at least 10.

That the number of imaginary roots of f must be greater than 8 follows from Theorem IV, and the proof is complete.

8. Applications and Examples. To illustrate the application of these theorems we may consider the following simple examples.

Example 1. A graph of the polynomial

$$3x^4 - 28x^3 + 72x^2 - 110$$

shows bend points at $x = 3$ and $x = 4$, whereas its greatest real root lies between 1.9 and 2. By Theorem III, we then have for its imaginary roots,

$$4 < a < 5.1, \quad 0 < b < 1.1.$$

Example 2. The equation

$$x^5 - 5x^4 + 10x^3 - 20x^2 + 40x - 26 = 0$$

has unity for its sole real root, as its derivative vanishes for $x = 2, 2$, and $\pm i\sqrt{2}$. Of the four imaginary roots, one pair must have its real part negative; and for the other pair we may write by Theorems I and II,

$$2 < a < 2.71, \quad 0 < b < 1.$$

Example 3. The equation

$$48x^5 - 85x^3 + 15x + 22 = 0$$

has unity for a double root, its remaining real root lying between -1.3 and -1.4 . Its derivative vanishes for $x = \pm 1$ and $\pm \frac{1}{4}$. By the corollary to Theorem IV, we have for the imaginary roots

$$-1.4 < a < 1, \quad 0 < b < 1.2.$$

When the roots of the polynomial itself are known, these theorems may often be useful in determining the nature of the roots of the derivative.

Example 4. The equation

$$f(x) \equiv (x + \frac{7}{8})(x - \frac{7}{8})[(x+1)^2 + 3]^2 \cdot [(x-1)^2 + 3]^2 = 0$$

has the real part of its imaginary roots outside of the interval between its least and greatest real roots. As the number of imaginary roots is not greater than eight, by Theorem IV the derivative can vanish only once within that interval. It cannot vanish on the boundary, as the real roots are not multiple. It cannot vanish outside the interval, for if $\rho > \frac{7}{8}$ were a real root of f' , by Theorem I we should have $\rho < 1$ and $b \leq \frac{1}{4}$, whereas, on the contrary, $b = \sqrt{3}$. Therefore the derivative can have but one real root; and by symmetry it is zero.

Example 5. The equation

$$f(x) = (x + \frac{7}{8})(x - \frac{7}{8})[(x + 1)^2 + 3]^6 \cdot [(x - 1)^2 + 3]^6 = 0$$

differs from that of Example 4 only in the multiplicity of its imaginary roots, but its derivative vanishes at least once between $-\frac{7}{8}$ and 0, and at least once between 0 and $+\frac{7}{8}$, as well as at the origin.

Example 6. The equation

$$x(x - 1)^{27}[(x + \frac{1}{18})^2 + \frac{275}{648}]^{60} = 0$$

has all its imaginary roots without and on the same side of the interval between its real roots; but because of the high multiplicity both of the imaginary roots and of the real root on the opposite side of the interval, the derivative vanishes at least once between 0 and $\frac{1}{4}$, and at least once between $\frac{1}{4}$ and 1, as well as at $x = \frac{1}{4}$.

RELATIONS ENTRE LES NOTIONS DE LIMITE ET DE DISTANCE*

PAR

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POSITION DU PROBLÈME

1. Un certain nombre de propriétés des ensembles linéaires peuvent être étendues aux ensembles abstraits, c'est à dire aux ensembles dont les éléments sont de nature quelconque ou inconnue.

C'est ainsi que G. Cantor a pu développer la théorie des nombres cardinaux et ordinaux qui s'applique aussi bien aux ensembles abstraits qu'aux ensembles linéaires. C'est ainsi qu'on peut donner une définition de l'intégrale d'une fonction étendue à un ensemble abstrait en généralisant d'une façon presque immédiate une définition donnée par J. Radon pour l'intégrale étendue à un espace à un nombre fini de dimensions† comme je l'ai montré récemment.‡

2. Mais si l'on veut aller plus loin, on est amené à introduire la notion de limite ou même la notion de distance tout en s'efforçant de les présenter sous une forme qui ne fasse pas intervenir la nature des éléments de la classe d'éléments considérée.

En ce qui concerne la notion de limite, on peut se contenter d'admettre qu'il existe une règle d'ailleurs quelconque ou inconnue permettant de décider si une suite infinie d'éléments $A_1, A_2, \dots, A_n, \dots$, de la classe considérée converge ou non vers un élément déterminé A . On se bornera à supposer que ce critère est tel que 1° si les éléments de la suite sont identiques, la suite converge et converge vers A_1 ; 2° si une suite $\{A_n\}$ converge vers A , toute suite infinie extraite de la suite $\{A_n\}$ converge aussi vers A .

De telles classes d'ensembles, que j'ai appelées classes (L), ne sont pas moins générales qu'une classe abstraite quelconque. Rien n'empêche en effet de considérer une classe abstraite arbitraire comme classe (L) en s'y prenant par exemple de la manière suivante: on porte son attention sur une suite arbitraire $A, A_1, A_2, \dots, A_n, \dots$. Une suite d'éléments distincts sera considérée comme convergente si c'est la suite $A_1, A_2, \dots, A_n, \dots$, ou si elle est extraite de cette suite, et dans les deux cas la suite est dite converger

* Presented to the Society, April, 1917.

† J. Radon, *Wiener Sitzungsberichte*, vol. 122, 2a (1913), pp. 1295-1438.

‡ M. Fréchet, *Bulletin de la Société Mathématique de France*, vol. 43 (1915), pp. 248-265.

vers l'élément A . Dans le cas contraire, la suite est considérée comme divergente.

Ceci étant et moyennant une extension convenable des définitions ordinaires, on peut développer une théorie de la connexion des ensembles abstraits dans les classes où une telle définition de la limite est adoptée* ou encore généraliser quelques théorèmes sur les fonctions également continues.†

3. Mais si l'on veut généraliser des théorèmes plus précis de la théorie des fonctions, une définition aussi indéterminée de la limite ne suffit plus. J'ai donc été amené, dans ma Thèse, à étudier l'effet de restrictions de plus en plus grandes apportées à l'idée de limite, ces restrictions restant toutefois telles que toutes les définitions classiques de la limite dans les ensembles les plus importants (ensembles de points, de courbes, de fonctions continues, etc.) y satisfont par avance.

La notion qui intervient de la façon la plus essentielle pour faciliter l'étude de la limite dans la théorie des ensembles linéaires est la *notion de distance*. Réduite à ses caractéristiques abstraites, on peut la décrire ainsi:

Dans une classe (L) , on peut définir la distance si à tout couple A, B d'éléments de la classe correspond un nombre $(A, B) = (B, A) \geq 0$ jouissant des propriétés suivantes:

- 1° (A, B) n'est nul que si A, B coïncident;
- 2° si, A restant fixe, la suite $\{B_n\}$ converge vers A , (A, B_n) doit converger vers zéro et réciproquement;
- 3° on a, quels que soient les éléments A, B, C de la classe,

$$(B, C) \leq (A, B) + (A, C).$$

J'appellerai un tel nombre la *distance de A à B* et je dirai d'une classe (L) où on peut définir une distance que *c'est une classe (D)* .‡

Il faut tout d'abord remarquer que les conditions 1°, 2°, 3° à elles seules ne peuvent apporter aucune limitation à l'arbitraire de la nature des éléments de la classe. Considérons en effet la catégorie la plus générale des classes d'éléments abstraits. On peut d'abord définir au moins d'une façon une nombre satisfaisant à 1°, 3°—il suffit par exemple de prendre pour (A, B) la valeur 1 quand A, B sont distincts, 0 quand ils coïncident.§ Une fois choisie une définition de la distance satisfaisant à 1° et 3°, si l'on définit précisément la limite par la condition 2°, on aura bien une classe (D) .

4. Le problème qui se pose n'est donc pas de savoir à quelle condition une classe abstraite est une classe (D) , mais à quelles conditions simples et

* M. Fréchet, *Mathematische Annalen*, vol. 68 (1910), pp. 145-168.

† M. Fréchet, *Thèse*, *Rendiconti del Circolo Matematico di Palermo*, vol. 22 (1906), pp. 1-74.

‡ Je modifie donc ici et dans la suite les dénominations adoptées dans ma Thèse. Ce que j'y ai appelé écart et voisinage je l'appellerai ici distance et écart uniformément régulier.

§ Voir Note Supplémentaire A à la fin de cette mémoire.

indépendantes une classe (L) où les suites convergentes et leurs limites sont déjà définies est-elle une classe (D) et comment on y peut calculer la distance (A, B).

Ainsi, l'objet de ce travail est le suivant: sachant qu'on peut toujours de bien des façons définir dans une classe d'éléments abstraits les suites convergentes et leurs limites, il s'agit de déterminer à quelles conditions supplémentaires il faut assujettir ce choix pour que l'on puisse définir sur cette classe une distance telle que la convergence définie d'avance ne soit pas modifiée quand on la définit au moyen de cette distance.

RÉDUCTION DU PROBLÈME

5. Pour résoudre ce problème, nous porterons d'abord notre attention sur les conditions 1° et 2°. Si dans une classe (L), on parvient à définir un nombre $(A, B) = (B, A) \geq 0$ satisfaisant aux conditions 1° et 2°, on dira que cette classe est une classe (E), ou encore qu'on a pu y définir un *écart*,* à savoir le nombre (A, B) , qui sera l'écart des éléments A et B . Si cet écart satisfait en même temps à la condition 3°, ce sera en même temps une "distance."

D'autre part il est facile de vérifier que dans une classe (D) tout ensemble dérivé est fermé, c'est à dire que si une suite d'éléments $\{A_n\}$ converge vers l'élément A et si chaque élément A_n est la limite d'une suite convergente $A_1^{(n)}, A_2^{(n)}, \dots, A_p^{(n)}, \dots$, on peut extraire de l'ensemble à double entrée des $A_p^{(n)}$ une suite $A_{p_1}^{(n_1)}, A_{p_2}^{(n_2)}, \dots$, qui converge vers A . Car

$$(A, A_{p_i}^{(n_i)}) \leq (A, A_{n_i}) + (A_{n_i}, A_{p_i}^{(n_i)}),$$

et en prenant n_i puis p_i assez grand, on rend le second membre et par suite le premier aussi petit que l'on veut.

Nous dirons d'une classe (L) où tout ensemble dérivé est fermé que c'est une classe (S).

6. Nous avons dès lors obtenu deux conditions nécessaires pour qu'une classe (L) admette une définition de la distance, c'est à dire soit une classe (D). Il faut que cette classe (L) soit une classe (E) et en même temps une classe (S).

Il est facile de voir que ces deux conditions sont indépendantes.

Considérons en effet la classe formée des points d'un plan définis comme il suit par leurs coordonnées cartésiennes:

pour le point A_0 : $x = 0, y = 0$;

pour un quelconque A_n des points $A_1, A_2, \dots, A_n, \dots$: $x = 1/n, y = 0$;

pour un quelconque $A_p^{(n)}$ des points de la suite $A_1^{(1)}, A_1^{(2)}, A_2^{(1)}, \dots$, à double entrée: $x = 1/n, y = 1/p$.

* J'introduis ici une notation (E) et un mot "écart" que j'ai employé dans ma Thèse, là où j'emploie maintenant la notation (D) et le mot distance.

Nous pouvons définir ainsi un écart de deux points quelconques de la classe; il sera pris égal:

à l'unité si l'un des points est A_0 et l'autre l'un des points $A_p^{(n)}$;

à la distance géométrique des deux points sur le plan dans tout autre cas. Ceci étant nous pouvons considérer la classe comme une classe (L) et en même temps comme une classe (E) en y définissant la limite précisément par la condition 2° énoncée plus haut.

Cependant elle ne peut être une classe S , bien évidemment.

Inversement une classe (S) peut ne pas être une classe (E) . En effet reprenons la même classe d'éléments, mais cette fois définissons autrement la convergence d'une suite. Une suite d'éléments de la classe sera dite convergente vers l'un d'eux lorsque leur distance géométrique à ce dernier tend vers zéro, à moins que la suite ne comprenne les deux points $A_1^{(1)}$, $A_1^{(2)}$, cas où la suite sera dite divergente. Il est évident que dans cette classe tout ensemble dérivé y est fermé. Pourtant, on ne peut y définir un écart sans altérer la convergence des suites. Car dans ce cas, si on considère la suite $A_1, A_2, \dots, A_n, \dots$, par exemple, elle converge vers A_0 , donc si l'écart existait, la suite $(A_0, A_1), (A_0, A_2), \dots, (A_0, A_n), \dots$, devrait converger vers zéro. Par suite, il en serait de même de la suite de nombres $(A_0, A_1^{(1)}), (A_0, A_1^{(2)}), (A_0, A_1), (A_0, A_2), \dots, (A_0, A_n), \dots$, et alors la suite de points $A_1^{(1)}, A_1^{(2)}, A_1, A_2, \dots, A_n, \dots$, convergerait vers A_0 contrairement à l'hypothèse.

7. Nous remarquons immédiatement que les exemples précédents montrent en même temps que *si l'on a défini d'avance la limite dans une classe abstraite, il n'est pas toujours possible d'y définir la distance de deux éléments quelconques de façon que la limite antérieurement définie puisse aussi être définie sans altération au moyen de la distance*. Nous voyons aussi qu'il n'est même pas toujours possible d'y définir un écart sans altérer les limites.

8. Nous pouvons en outre décomposer en plusieurs autres conditions indépendantes la condition pour une classe (L) d'être une classe (E) .

Si une classe est (E) , la condition nécessaire et suffisante pour qu'une suite d'éléments $\{A_n\}$ converge vers l'élément A est que pour q donné on puisse trouver p tel que

$$(A_n, A) < \frac{1}{q}$$

pour $n > p$. Autrement dit si l'on appelle Σ_q l'ensemble des éléments B tels que $(A, B) < 1/q$, il faut et il suffit que quel que soit q , la suite des A_n soit contenue dans Σ_q à partir d'un certain rang variable avec q .

Remarquons d'ailleurs que Σ_{q+1} fait partie de Σ_q et que $\Sigma_1, \Sigma_2, \dots$ ont un seul élément commun savoir A .

On est alors amené à la définition suivante. Soit une classe (L) ; on dira

que c'est une classe (V) ou encore qu'on peut y définir un voisinage* lorsque la condition suivante est réalisée:

Pour tout élément A , il existe une suite d'ensembles $U_1, U_2, \dots, U_q, \dots$ telle que la condition nécessaire et suffisante pour qu'une suite $\{A_n\}$ converge vers A est que quel que soit q , la suite $\{A_n\}$ fasse partie de l'ensemble U_q à partir d'un certain rang (variable en général avec q).

On remarque aussitôt, que s'il en est ainsi l'élément A est le seul élément commun à $U_1, U_2, \dots, U_q, \dots$, et qu'en appelant T_q l'ensemble $U_1 \cdot U_2 \cdot \dots \cdot U_q$ des éléments communs à U_1, U_2, \dots, U_q , l'ensemble T_q est contenu dans T_{q-1} et la suite des ensembles $T_1, T_2, \dots, T_q, \dots$, peut remplacer la suite des ensembles $U_1, U_2, \dots, U_q, \dots$.

Ainsi on peut supposer que U_q appartienne à U_{q-1} et on doit se souvenir que A est le seul élément commun aux U_q . Lorsqu'il y a lieu de distinguer les U_q relatifs à un élément A de ceux relatifs à un élément B , on désignera les premiers par $U_q^{(A)}$ et les seconds par $U_q^{(B)}$.

Nous pourrions donc dire qu'un élément B est très voisin de A lorsque B est dans un ensemble $U_q^{(A)}$ d'indice très grand. Ou encore nous pouvons dire que les ensembles $U_1^{(A)}, U_2^{(A)}, \dots$, constituent successivement des voisinages de plus en plus étroits de A .

9. Nous avons alors à résoudre la question suivante: soit une classe (L) où les suites convergentes et leurs limites sont déterminées d'avance, est-il toujours possible d'assigner à chaque élément A une suite d'ensembles $T_1^{(A)}, T_2^{(A)}, \dots$, déterminant des voisinages de plus en plus étroits de A , de sorte que les suites convergentes et leurs limites ne soient pas modifiées?

Nous avons aussi à résoudre la question suivante: si dans une classe (V) le voisinage autour de chaque élément est défini d'une certaine façon, est-il toujours possible d'assigner un écart à tout couple d'éléments de l'ensemble, de sorte que la convergence et la limite éventuelle d'une suite ne soient pas modifiées, qu'on définisse celles-ci par le moyen du voisinage ou par celui de l'écart?

Nous allons voir que la réponse à ces deux questions est négative et nous allons déterminer quelles sont les conditions supplémentaires que doit vérifier la définition de la limite pour qu'on puisse la définir au moyen soit du voisinage, soit de l'écart.

10. La réponse à la première question est négative, c'est à dire qu'une classe (L) étant donnée, on ne peut pas toujours y définir un voisinage* sans altérer la convergence ou la limite de ses suites d'éléments.

Pour qu'une classe (L) puisse être considérée comme une classe (V), sans modifier la convergence des suites d'éléments ni les limites des suites convergentes, il faut et il suffit que:

* J'emploie ici la notation (V) et le mot de voisinage dans un sens plus général que celui que j'avais adopté dans ma Thèse. Voir aussi Note Supplémentaire B à la fin de cette mémoire.

1° toute suite convergente d'éléments de la classe (L) reste convergente quand on lui ajoute un nombre fini d'éléments de la classe;

2° pour tout élément non isolé A de la classe, toute suite σ d'éléments $A_1^{(\sigma)}, A_2^{(\sigma)}, \dots$, convergeant vers A et tout entier n , il existe un entier $q_n^{(\sigma)}$ jouissant de la propriété suivante: Soient $\sigma_1, \sigma_2, \dots$, une suite de suites (distinctes ou non) convergeant vers A et n_1, n_2, \dots , une suite d'entiers. Si quel que soit l'entier N on peut trouver l'entier i_0 tel que pour $i > i_0$, $n_i > q_N^{(\sigma)}$, alors la suite $A_{n_1}^{(\sigma_1)}, A_{n_2}^{(\sigma_2)}, A_{n_3}^{(\sigma_3)}, \dots$, converge aussi vers A .

Ces conditions sont nécessaires. Car si la classe est (V), la limite A d'une suite A_1, A_2, \dots , est définie de la façon suivante. Il existe des ensembles $T_1, T_2, \dots, T_r, \dots$, chacun dans le précédent, n'ayant en commun que l'élément A , tels que si A_1, A_2, \dots , converge vers A , cette suite est contenue dans chaque T_r à partir d'un certain rang. S'il en est ainsi, cette condition sera encore réalisée si on ajoute un nombre fini d'éléments de la classe à la suite, donc la convergence et la limite de cette suite n'en seront pas modifiées. D'autre part soit $q_n^{(\sigma)}$ le rang du premier élément de la suite $A_1^{(\sigma)}, A_2^{(\sigma)}, \dots, A_n^{(\sigma)}, \dots$, à partir duquel les termes de cette suite supposée convergeant vers A , sont tous compris dans T_n . Et supposons que quel que soit N on peut trouver i_0 tel que $n_i > q_N^{(\sigma)}$ pour $i > i_0$. Alors $A_{n_i}^{(\sigma)}$ est dans T_N pour $i > i_0$; autrement dit, quel que soit N , la suite $A_{n_1}^{(\sigma_1)}, A_{n_2}^{(\sigma_2)}, \dots$, appartient à T_N à partir d'un certain rang i_0 variable avec N . Elle est donc convergente et sa limite est A .

Inversement, les conditions 1° et 2° sont suffisantes pour que la classe (L) soit une classe (V).

En effet; tout élément B de la classe appartient à au moins une suite convergeant vers A (par exemple $B, A, A, A, \dots, A, \dots$); soit $A_1^{(\sigma)}, A_2^{(\sigma)}, \dots, A_{p-1}^{(\sigma)}, B, A_{p+1}^{(\sigma)}, \dots$, cette suite. Nous appellerons $U_n^{(A)}$ l'ensemble des éléments B tels que dans une au moins des suites σ tendant vers A et comprenant B celui-ci occupe un rang supérieur à $q_n^{(\sigma)}$. Les ensembles $U_n^{(A)}$ jouissent alors des propriétés voulues. Considérons d'abord une suite $A_1^{(\sigma)}, A_2^{(\sigma)}, \dots, A_p^{(\sigma)}, \dots$, convergeant vers A . Quel que soit n , l'élément $A_p^{(\sigma)}$ occupera pour p assez grand ($p \geq p_0$) un rang supérieur à $q_n^{(\sigma)}$. Donc les éléments de cette suite à partir de A_{p_0} appartiennent à $U_n^{(A)}$. Inversement si les éléments de cette suite appartiennent quel que soit N à $U_N^{(A)}$ à partir d'un certain rang p_0 (variable avec N) c'est que pour $p \geq p_0$, $A_p^{(\sigma)}$ appartient à au moins une suite (σ_p) convergeant vers A où il occupe un rang $r_p > q_N^{(\sigma)}$. Autrement dit, étant donnée la suite

$$A_1^{(\sigma)} = A_{r_1}^{(\sigma_1)}, \quad A_2^{(\sigma)} = A_{r_2}^{(\sigma_2)}, \quad \dots, \quad A_p^{(\sigma)} = A_{r_p}^{(\sigma_p)}, \quad \dots,$$

on peut pour chaque valeur de N trouver p_0 , tel que $p \geq p_0$ entraîne $r_p > q_N^{(\sigma)}$ et alors d'après la condition 2° la suite précédente converge vers A .

11. *Les conditions 1° et 2° sont d'ailleurs indépendantes.*

On peut satisfaire à 2° sans satisfaire à 1° en prenant par exemple comme classe les points d'un segment BC extrémités comprises et en y définissant la limite comme d'ordinaire par la distance sauf que les suites comprenant le point B et dont la distance des points à B ne tend pas vers zéro seront toujours déclarées divergentes dans cette classe. Alors si on considère une suite convergeant vers A et si on lui adjoint B , elle cesse d'être convergente; 1° cesse d'être vérifié. Au contraire 2° est encore vérifié. Soit en effet une suite $A_1^{(\sigma)}, A_2^{(\sigma)}, \dots$, convergeant vers A . On appellera $q_n^{(\sigma)}$ le rang à partir duquel on a constamment $AA_p^{(\sigma)} < 1/n$. Alors si quel que soit N , on peut trouver i_0 , tel que pour $i > i_0$, on ait $n_i > q_N^{(\sigma)}$, c'est que $AA_{n_i}^{(\sigma)} < 1/N$ pour $i > i_0$. Si $A \neq B$, aucune des suites σ_i , supposées convergeant vers A , ne contiendra B ; par suite la suite $A_{n_1}^{(\sigma)}, A_{n_2}^{(\sigma)}, \dots$, non plus, et alors elle convergera vers A dans la classe. Si $A = B$, la présence de B dans les suites n'offre plus de difficulté.

On peut aussi satisfaire à 1° sans satisfaire à 2°.

Nous allons donner un exemple d'une classe (L) où la limite est définie de façon à satisfaire aux trois conditions 1°, 2° du n° 3 et 1° du n° 10 sans qu'on puisse y définir un voisinage non seulement au sens de ma Thèse mais même au sens plus large adopté ici (n° 8). Il suffit de montrer que la condition 2° du n° 10 n'est pas satisfaite.

Cet exemple est celui d'une classe (L) dont les éléments sont les fonctions $f(x)$ uniformes sur le segment J ($0 \leq x \leq 1$) et où la limite est définie comme d'ordinaire. C'est à dire que $f_n(x)$ converge vers $f(x)$ si ceci a lieu en chaque point de l'intervalle (uniformément ou non). Alors les trois conditions mentionnées plus haut sont satisfaites. Il faut montrer que la dernière condition 2° ne l'est pas.*

Prenons pour simplifier le cas où l'élément A est la fonction zéro (c'est à dire égale à zéro en tout point de l'intervalle J). Montrons que si pour chaque entier N et pour chaque suite σ de fonctions convergeant vers zéro, $f_1^\sigma(x), f_2^\sigma(x), \dots$, on détermine un certain entier q_N^σ , aucun choix de l'ensemble des entiers q_N^σ ne satisfera à la condition 2°. Considérons en particulier la suite σ_n formée des fonctions $f_1^n(x), f_2^n(x), \dots, f_p^n(x), \dots$, de la manière suivante. La fonction $y = f_p^n(x)$ est de période $1/2^n$ et elle est représentée par une ligne brisée dont les premiers sommets sont les points de coordonnées

$$(0, 0); \quad \left(\frac{1}{2^{n+p+1}}, 2\right); \quad \left(\frac{1}{2^{n+p}}, 0\right); \quad \left(\frac{1}{2^n}, 0\right).$$

On voit que les éléments de la suite σ_n convergent vers zéro quand p croît indéfiniment. À σ_n et N correspond un certain entier $q_N^{\sigma_n}$; posons

* Voir Note Supplémentaire C à la fin de cette mémoire.

$$p_n = 1 + q_1^{\sigma_n} + q_2^{\sigma_n} + \dots + q_n^{\sigma_n}.$$

Extrayons des suites σ_n la suite σ :

$$f_{p_1}^1, f_{p_2}^2, \dots, f_{p_n}^n, \dots.$$

Pour chaque valeur de N on peut trouver i tel que pour $n > i$ on ait $p_n > q_N^{\sigma_n}$. Il suffit de prendre $i = N$, car alors

$$p_n = 1 + q_1^{\sigma_n} + \dots + q_N^{\sigma_n} + \dots + q_n^{\sigma_n} > q_N^{\sigma_n}.$$

Si donc la condition 2° était vérifiée, la suite σ convergerait vers zéro.

Or on peut facilement trouver un point x_0 où $f_{p_n}^n(x)$ ne tend pas vers zéro. En effet, on a évidemment

$$f_p^n(x) \geq 1 \quad \text{pour} \quad 0 \leq \frac{r}{2^n} + \frac{1}{2^{n+p+2}} \leq x \leq \frac{r}{2^n} + \frac{3}{2^{n+p+2}} \leq 1,$$

quand r est un entier quelconque prenant l'une des valeurs $0, 1, 2, \dots, 2^n - 1$. Appelons $I_{n,p}^r$ l'intervalle précédent. Il suffit alors de prouver qu'il existe un point x_0 commun à une infinité des intervalles $I_{1,p_1}^1, I_{2,p_2}^2, \dots, I_{n,p_n}^n, \dots$. Or étant donné l'intervalle I_{n,p_n}^r , prenons $n' = n + p_n + 2$, puis r' tel que

$$\frac{r_n}{2^n} + \frac{1}{2^{n+p_n+2}} = \frac{r'}{2^{n'}} < \frac{r' + 1}{2^{n'}} < \frac{r_n}{2^n} + \frac{3}{2^{n+p_n+2}}.$$

Alors $I_{n',p_{n'}}^{r'}$ est compris dans I_{n,p_n}^r . On a donc une succession d'intervalles pris dans la suite des I_{n,p_n}^r et chacun compris dans le précédent. Ils auront donc un point commun x_0 .

12. La réponse à la seconde question du n° 9 est aussi négative. Nous allons en effet donner l'exemple d'une classe (V) d'éléments où la limite est définie au moyen du voisinage et ne peut être définie sans altération au moyen d'un écart.

Considérons en effet la classe (V) formée des points d'un segment ab et où le voisinage d'un point d'abscisse x_0 est défini de la façon suivante. On appellera $T_0^{x_0}$ l'ensemble des points du segment ab et $T_n^{x_0}$ (pour n entier ≥ 1) l'ensemble des points du segment de longueur $1/n$ ayant x_0 pour extrémité droite. Chacun de ces ensembles $T_n^{x_0}$ contient x_0 et contient le suivant $T_{n+1}^{x_0}$.

On voit évidemment que dans cette classe pour qu'une suite $\{x_p\}$ converge vers x_0 , il faut et il suffit que la distance géométrique $x_0 x_p$ tende vers zéro et de plus que les points x_p soient à partir d'un certain rang à gauche de x_0 (ou confondus avec x_0).

Il s'agit de montrer qu'aucune définition d'un écart (x, x') dans cette classe ne peut conduire à la même définition de la convergence.

En effet supposons une telle définition possible. Alors si x' est un point arbitraire à droite d'un point fixe x l'écart (x, x') aura une borne inférieure positive. En effet, dans le cas contraire, cette borne serait nulle et on pourrait, quel que soit n , trouver un point x_n à droite de x tel que $(x, x_n) < 1/n$. Par suite la suite $\{x_n\}$ située à droite de x , devrait converger vers x dans la

classe, ce qui est contraire à la définition actuelle de la limite. Soit donc $F(x)$ la borne inférieure positive considérée, bien déterminée pour chaque valeur de x dans ab .

Pour tout point x_0 de ab et toute valeur de l'entier p , on peut déterminer un intervalle $x'_0 x_0$ limité à droite à ce point x_0 et à l'intérieur duquel $F(x) < 1/p$. Car il suffit à cet effet de déterminer le point x'_0 de sorte que pour $x'_0 < x < x_0$, on ait $(x, x_0) < 1/p$. Prenons en particulier $x_0 = b$; on a un intervalle $x''_0 b$ dans lequel $F(x) < 1/p$. Soit α la borne inférieure des nombres x''_0 telle que dans l'intervalle $x''_0 b$, on ait $F(x) < 1/p$, sauf peut être en un ensemble dénombrable de points (il y a au moins un de ces intervalles, à savoir $x''_0 b$). On a nécessairement $\alpha = a$. Car si $a < \alpha$, il suffirait de former à gauche de α comme plus haut un intervalle $\beta\alpha$ dans tout l'intérieur duquel $F(x) < 1/p$. Et alors l'intervalle $\beta b = \beta\alpha + \alpha b$ serait un intervalle plus grand que αb , où $F(x) < 1/p$ sauf en un ensemble dénombrable de points.

Mais alors on voit que $F(x)$ ne peut être $\geq 1/p$ qu'en un ensemble dénombrable E_p de points de ab . Par suite $F(x)$ ne pourrait être positif que dans l'ensemble dénombrable de points qui est la somme des ensembles dénombrables $E_1 + E_2 + \dots + E_p + \dots$, et non en tout point de ab .

13. On peut remarquer que dans le cas d'une classe (V) , on pourrait déterminer une sorte d'écart dissymétrique. Appelons en effet $((A, B))$ l'inverse de la plus haute valeur de n telle que le voisinage T_n^A contienne B . Ce nombre est ≥ 0 , fini et bien déterminé si A et B sont distincts. Nous poserons $((A, B)) = 0$ si A et B coïncident. On voit qu'en général on aura $((A, B)) \neq ((B, A))$. Cependant, A étant un élément fixe, si B_p tend vers A , $((A, B_p))$, — mais non nécessairement $((B_p, A))$, — tend vers zéro, et réciproquement.

14. Étant prouvé qu'une classe (V) n'est pas nécessairement une classe (E) , nous allons déterminer maintenant à quelle condition il en est ainsi.

Nous supposons donc que dans une classe (V) , la limite soit définie en attachant à chaque élément A une suite d'ensembles $T_1^{(A)}, T_2^{(A)}, \dots, T_n^{(A)}, \dots$, définissant des voisinages de plus en plus étroits. Et nous cherchons à quelle condition doivent satisfaire ces ensembles $T_n^{(A)}$ pour que la limite ainsi définie puisse être définie sans altération de la convergence ni de la limite par le moyen d'un écart (A, B) attribué à chaque couple d'éléments A, B, C , cet écart satisfaisant aux conditions 1°, 2° du n° 10.

Cette condition est la suivante: il faut et il suffit que l'on puisse attacher à chaque élément A une suite d'entiers non décroissants et qui tendent vers l'infini

$$1 = r_1^A \leq r_2^A \leq \dots \leq r_n^A \leq \dots,$$

choisis de sorte que si l'on se donne arbitrairement un entier N et un élé-

ment A de la classe, on puisse déterminer un entier m pour lequel l'ensemble $T_{r_N^B}^{(B)}$ contient nécessairement l'élément A , si B appartient à $T_m^{(A)}$.

La condition est nécessaire. En effet, soit une classe (V) où la limite peut aussi être déterminée par le moyen d'un écart (A, B) .

Comme on peut supposer que $T_1^{(A)}$ contient toute la classe, alors, pour chaque valeur de n , l'un au moins des ensembles $T_1^{(A)}, T_2^{(A)}, \dots$, contient tous les éléments B tel que $(A, B) < 1/n$. Si $T_n^{(A)}$ est précisément l'un de ces ensembles nous poserons $r_n^A = n$; sinon nous appellerons r_n^A le rang le plus haut ($< n$) de tous ces ensembles.

On voit que dans tous les cas $T_{r_n^A}^{(A)}$ contient tous les éléments B tels que $(B, A) < 1/n$ et que $r_n^A \leq r_{n+1}^A$. On voit aussi que r_n^A tend vers l'infini quand A est fixe; sans quoi il aurait une borne supérieure finie k . Et alors quel que soit $n > k$ il y aurait en dehors de $T_{k+1}^{(A)}$ un élément B_n tel que $(B_n, A) < 1/n$. Mais alors si la limite est définie par l'écart, B_n tendrait vers A , et si elle est définie par le voisinage, B_n ne tendrait pas vers A .

Ceci étant donnons nous arbitrairement N et A . On peut choisir un entier m tel que pour tout élément B de $T_m^{(A)}$, on ait $(B, A) < 1/N$. (Car dans le cas contraire, il y aurait dans chaque ensemble $T_m^{(A)}$ un élément C_m tel que $(C_m, A) > 1/N$. Alors selon qu'on emploierait le voisinage ou l'écart C_m tendrait ou ne tendrait pas vers A .)

Or si $(B, A) < 1/N$, $T_{r_N^B}^{(B)}$ contient A . La condition est bien nécessaire.

Inversement, supposons que dans une classe (V) l'existence des suites r_n^A satisfaisant aux conditions demandées soit assurée. Montrons qu'on peut aussi définir la limite dans la classe, sans altérer la convergence, au moyen d'un écart.

Pour cela appelons q_B^A la plus grande valeur de n telle que $T_{r_n^A}^{(A)}$ contient B . Il y a au moins une telle valeur de n puisque $T_{r_1^A}^{(A)} = T_1^{(A)}$ est formé de toute la classe. Alors q_B^A est un nombre bien déterminé et fini si A et B sont distincts.

Posons alors $(A, B) = (B, A) = 0$ si A et B coïncident et

$$(A, B) = (B, A) = \frac{1}{q_B^A} + \frac{1}{q_A^B}$$

dans le cas où ils sont distincts. Alors on a défini pour tout couple d'éléments un nombre ≥ 0 qui n'est nul que si les deux éléments coïncident. Il s'agit de montrer que si une suite B_p tend vers A , (B_p, A) tend vers zéro et réciproquement.

Tout d'abord si (B_p, A) tend vers zéro, $q_{B_p}^A$ croît indéfiniment; donc, si N est donné, on a pour p assez grand ($p > m$): $q_{B_p}^A > N$; donc $T_{r_N^A}^{(A)}$ contient B_p , par définition de $q_{B_p}^A$. Mais si on s'est donné A et un entier α , on peut choisir N pour que $r_N^A > \alpha$ et par suite $T_\alpha^{(A)}$ contient aussi B_p à partir du rang m . Autrement dit, puisque α est arbitraire, B_p tend vers A .

Inversement si B_p tend vers A , et si α est un entier donné, B_p est dans $T_{r_A}^{(A)}$ à partir d'un certain rang k ; donc $q_{B_p}^A > \alpha$ pour $p > k$. Donc $q_{B_p}^A$ tend vers l'infini avec p . Pour montrer que (A, B_p) tend vers zéro, il suffit donc de montrer que $q_A^{B_p}$ tend aussi vers l'infini. Or dans le cas contraire on pourrait trouver un nombre positif λ tel que pour une infinité de valeurs de p , on ait $q_A^{B_p} < \lambda$. Mais pour une telle valeur de p , $T_{r_A}^{(B_p)}$ ne contiendrait pas A . Or par hypothèse, étant donnés A et λ on peut trouver un entier m tel que si B_p appartient à $T_m^{(A)}$, $T_{r_A}^{(B_p)}$ contienne A . Il faudrait donc que pour une infinité de valeurs de p , B_p n'appartienne pas à $T_m^{(A)}$, ce qui est contraire à la définition de la limite par le voisinage.

15. Pour élucider complètement les relations entre les notions de limite et de distance, il reste à résoudre le problème suivant:

Nous avons vu que pour qu'une classe (L) soit une classe (D) , il faut qu'elle soit (S) et (E) . Ces deux conditions nécessaires, qui sont indépendantes, sont-elles suffisantes? Et dans le cas de la négative, quelle est la nature de la condition supplémentaire?

Ce problème n'est pas résolu dans le présent mémoire. Nous nous contenterons des indications suivantes.

16. On peut sérier les difficultés en passant du cas des classes qui sont à la fois (E) et (S) au cas des classes (D) par l'intermédiaire des classes que j'ai appelées (V) dans ma Thèse, notation à laquelle je n'attache plus ici le même sens, le terme voisinage se trouvant plus adéquatement utilisé dans le présent mémoire. Nous appellerons ces classes non plus (V) , mais (E_r) , et nous dirons que l'écart dans ces classes est *uniformément* régulier. Les considérations suivantes vont expliquer la raison de ces dénominations et pourquoi les classes (E_r) s'introduisent naturellement dans la solution du problème que nous venons de poser.

17. Voyons donc quelles seraient les propriétés d'une classe qui serait à la fois (E) et (S) . Les conditions 1° et 2° vérifiées par une classe D restant remplies par une telle classe, il s'agit de savoir ce qui remplace la condition 3°. A cet effet démontrons le théorème suivant:

*La condition nécessaire et suffisante pour qu'une classe (E) soit (S) , (autrement dit pour que tout ensemble dérivé soit fermé dans une classe où la limite est définie à l'aide de l'écart) est qu'à tout élément A et à tout nombre positif δ , on puisse faire correspondre un nombre η tel que si B est un élément déterminé dont l'écart avec A soit moindre que η et si ω est un nombre positif convenablement choisi une fois connus δ , η , A , B , alors pour tout élément C dont l'écart avec B est moindre que ω , on aura**

$$(A, C) < \delta.$$

* On peut exprimer brièvement cette condition en disant que si (A, B) et (B, C) sont des infiniments petits, (A, C) est aussi un infiniment petit. Mais la convergence n'est pas nécessairement uniforme.

En effet supposons d'abord que la classe soit (E) et (S) . Si le théorème n'était pas vrai pour un élément A de l'ensemble et un certain nombre $\delta > 0$, la condition $(A, C) < \delta$ ne serait pas nécessairement vérifiée et ceci quelle que soit la valeur adoptée pour η . En particulier si l'on prenait $\eta = 1/n$, il y aurait au moins un élément B_n tel que $(A, B_n) < 1/n$, pour lequel aucun choix convenable de ω ne saurait assurer la condition recherchée. C'est à dire que si l'on prenait par exemple $\omega = 1/p$, il y aurait un élément $C_p^{(n)}$ tel que

$$(B_n, C_p^{(n)}) < \omega = \frac{1}{p} \quad \text{et} \quad (A, C_p^{(n)}) > \delta.$$

Mais, quand n restant fixe p croît indéfiniment, $C_p^{(n)}$ tend vers B_n . Et quand n croît indéfiniment, B_n tend vers A . L'ensemble dérivé de l'ensemble des $C_p^{(n)}$ comprend donc l'ensemble des B_n . Et comme il est fermé par hypothèse, il comprend aussi la limite A des B_n . Or la condition $(A, C_p^{(n)}) > \delta > 0$ montre que A ne peut être limite d'une suite d'éléments $C_p^{(n)}$.

Réciproquement supposons que pour une classe (E) la condition énoncée plus haut soit vérifiée. Alors tout ensemble E' dérivé d'un ensemble quelconque d'éléments de la classe est fermé. Autrement dit, si A est la limite d'une suite d'éléments $B_1, B_2, \dots, B_n, \dots$, de l'ensemble dérivé E' , A appartient à E' . En effet, chaque élément B_n de E' est la limite d'une suite d'éléments $C_1^{(n)}, C_2^{(n)}, \dots$. Pour δ donné, on peut trouver un nombre η tel que si $(A, B_n) < \eta$, il correspond à B_n un nombre ω_n pour lequel la condition $(B_n, C_p^{(n)}) < \omega_n$ entraîne $(A, C_p^{(n)}) < \delta$. Or quel que soit η on peut prendre n assez grand pour que $(A, B_n) < \eta$, et quel que soit ω_n , on peut prendre pour n fixe p assez grand pour que $(B_n, C_p^{(n)}) < \omega_n$. Les $C_1^{(n)}, C_2^{(n)}, \dots$ étant distincts ainsi que les B_1, B_2, \dots , on voit qu'on peut finalement faire correspondre à toute valeur $1/i$ de δ deux entiers n_i, p_i , tels que les éléments $C_{p_1}^{(n_1)}, C_{p_2}^{(n_2)}, \dots$ soient distincts et que $(A, C_{p_i}^{(n_i)}) < 1/i$. Autrement dit, A appartient aussi à E' .

18. Si le nombre ω ne dépend pas de l'élément B , alors en prenant pour ϵ le plus petit des nombres η et ω , on voit qu'à tout élément A et à tout nombre $\delta > 0$, on peut faire correspondre un nombre ϵ tel que si B, C sont deux éléments satisfaisant à la condition

$$(A, B) + (B, C) < \epsilon, \quad \text{et par suite à} \quad (A, B) < \eta, \quad (B, C) < \omega,$$

on a toujours

$$(A, C) < \delta.$$

Appelons $\psi(\theta, A)$ la borne supérieure, finie ou non, de (A, C) quand B, C sont des éléments quelconques tels que $(A, B) + (B, C) = \theta > 0$. On voit alors que $\psi(\theta, A)$ est un nombre > 0 qui tend vers zéro quand θ tend

vers zéro, A restant fixe. Car quel que soit le nombre $\delta > 0$ on peut trouver un nombre θ tel que si $(A, B) + (B, C) < \epsilon$, on a $(A, C) < \delta$, et par suite si $\theta < \epsilon$, $\psi(\theta, A) \leq \delta$. Et on a quels que soient B et C

$$(A, C) \leq \psi((A, B) + (B, C), A).$$

Si maintenant on suppose que $\psi(\theta, A)$ est aussi indépendant de A , on obtient une classe qui est ce que nous appellerons une *classe* (E_r), où on peut définir un écart *uniformément régulier* tel que l'on ait, quels que soient A, B, C ,

$$(A, C) \leq \psi((A, B) + (B, C)).$$

On voit que les classes (D), celles où la distance est définie, correspondent au cas particulier des classes (E_r) où $\psi(\theta) = \theta$. Il faut remarquer d'ailleurs qu'on a toujours $\psi(\theta) \geq \theta$ comme on le voit en prenant C identique à B .

Ces classes (E_r) ne sont autres que celles que j'ai appelées classes (V) dans ma Thèse. On voit en effet que si l'on appelle $\phi(\epsilon)$ une fonction partout supérieure à $\psi(2\epsilon)$ mais tendant vers zéro, on aura

$$(A, C) < \phi(\epsilon) \quad \text{pour} \quad (A, B) < \epsilon, (B, C) < \epsilon.$$

NOTES SUPPLÉMENTAIRES.*

A. Pour obtenir des distances infiniment petites, il suffirait de faire correspondre la classe à un ensemble fini linéaire non isolé avec conservation de la distance par définition. Et au cas où la puissance de la classe est supérieure au continu, de donner aux distances aboutissant aux éléments restants une valeur finie suffisamment grande.

B. Dans une note récente aux *Comptes-Rendus de l'Académie des Sciences* (Septembre, 1917) je généralise une nouvelle fois en abandonnant la condition que les ensembles U forment une suite dénombrable.

C. Il en résultera, en particulier, que *dans cette classe si importante on ne peut définir un écart*. On peut, d'ailleurs, donner de ce fait démonstrations directes plus simples. J'ai aussi fait remarquer, dans ma Thèse, que cette classe n'est pas une classe (s).

* These notes were received from the author too late to insert with the pages to which they belong.—THE EDITORS.

ON THE FOUNDATIONS OF THE CALCUL FONCTIONNEL OF FRÉCHET*

BY

A. D. PITCHER AND E. W. CHITTENDEN

In his thesis,† Fréchet gave a very beautiful generalization of the theory of point sets and of the theory of real valued functions of a real variable. His functions are real valued but the range of the independent variable is an abstract class Ω of elements q . He secures his results principally through the medium of a properly conditioned distance function δ , a generalization of the distance between two points, which associates with each pair $q_1 q_2$ of elements a real non-negative number $\delta(q_1 q_2)$. He is thus enabled to secure the more important theorems of point set theory and of real function theory, especially those relating to continuous functions and their properties. This theory of Fréchet has excited considerable interest and has received much attention from mathematicians. Various contributions to its foundations and to its content have been made.

In the present paper we follow the example of Fréchet in assuming once for all that $\delta(qq) = 0$ and that $\delta(q_1 q_2) = \delta(q_2 q_1)$. In other words we assume that the distance from an element to itself is zero and that the distance between two elements is independent of the order in which they are taken. In the first part of the paper we give very simple conditions on systems $(\Omega; \delta)$ which are sufficient for many purposes and which, in the case of compact sets, we show to be equivalent, so far as limit of a sequence is concerned, to the *voisinage* and thus to the *écart* of Fréchet.‡ The remainder of the paper is devoted to the theory of functions on the sets Ω of systems $(\Omega; \delta)$. In terms of the conditions already mentioned we generalize the results of Fréchet and Hahn as to the existence of non-constant continuous functions. We give two very

* The results of this paper have been presented to the Society at various times, especially April 2, 1915, and December 28, 1916.

† *Sur quelques points du calcul fonctionnel*, reprinted in *Rendiconti del Circolo Matematico di Palermo*, vol. 22 (1906), pp. 1-74.

See also, *A contribution to the foundations of Fréchet's calcul fonctionnel*, by T. H. Hildebrandt, *American Journal of Mathematics*, vol. 34 (1912), pp. 237-290.

‡ E. W. Chittenden has proved that the *voisinage* and the *écart* of Fréchet are equivalent so far as the limit of a sequence is concerned. Cf. these *Transactions*, vol. 18 (1917), pp. 161-166.

mild conditions on systems $(\Omega; \delta)$ which are completely independent and which secure for all continuous functions on Ω the important properties of the continuous functions of a real variable. Also we give a set of three completely independent conditions (the two above and one other) which we show to be both necessary and sufficient for a so-called *uniformly proper* theory of continuous functions where the term uniformly proper is used in a sense likely to be admitted by any one who gives the matter careful consideration.

1. INTRODUCTION

By the notation $(\Omega; \delta)$ we denote a set Ω of elements q and a function δ which assigns to each pair $q_1 q_2$ of elements a real number $\delta(q_1 q_2) \geq 0$. We also assume, once for all, that for every element q , $\delta(qq) = 0$, and that for every pair q_1, q_2 of elements $\delta(q_1 q_2) = \delta(q_2 q_1)$. We follow Fréchet in saying that q is a limit of the sequence q_n , $L_n q_n = q$, when and only when $L_n \delta(q_n q) = 0$. We shall be interested in the following properties of δ , or of systems $(\Omega; \delta)$.

- (1) If $\delta(q_1 q_2) = 0$ then $q_1 = q_2$.
- (2) If $L_n q_{1n} = q$ and $L_n \delta(q_{1n} q_{2n}) = 0$ then $L_n q_{2n} = q$.
- (3) If $L_n q_{1n} = q = L_n q_{2n}$ then $L_n \delta(q_{1n} q_{2n}) = 0$.
- (4) If $L_n \delta(q_{1n} q_{2n}) = 0$ and $L_n \delta(q_{2n} q_{3n}) = 0$ then $L_n \delta(q_{1n} q_{3n}) = 0$.
- (5) There is a function $\phi(e)$ such that $L_{e \rightarrow 0} \phi(e) = 0$ and such that if $\delta(q_1 q_2) \leq e$, $\delta(q_2 q_3) \leq e$ then $\delta(q_1 q_3) \leq \phi(e)$.
- (6) $\delta(q_1 q_2) + \delta(q_2 q_3) \geq \delta(q_1 q_3)$.

It will be seen at once that (2), (3), and (4) are important properties which are implied by (5) or (6). We will show that (2), (3), and (4) play a fundamental rôle.

The notation δ^n denotes the fact that δ has the property (n). Thus δ^{13} denotes a δ which possesses the properties (1) and (3). The *voisinage* of Fréchet is a δ^{15} and the *écart* a δ^{16} .

Our terminology is that of Fréchet except that we do not wish to imply that the limit of a sequence is unique. Thus a set $\bar{\Omega}$ of the set Ω of a system $(\Omega; \delta)$ is said to be *compact* in case every sequence of distinct elements of $\bar{\Omega}$ gives rise to at least one limiting element. $\bar{\Omega}$ is *closed* in case every limiting element of $\bar{\Omega}$ is of $\bar{\Omega}$. It should be noted that, in a system $(\Omega; \delta)$ where $\delta(q_1 q_2)$ may be zero for q_1 and q_2 distinct, an element q may be the limit of a sequence composed of a single element, other than q itself, repeated infinitely often. If $\bar{\Omega}$ is compact and closed then $\bar{\Omega}$ is said to be *extremal*. For the case of non-unique limits it is desirable to take special note of classes $\bar{\Omega}$ which may not be closed but which are such that every sequence $\{q_n\}$ of $\bar{\Omega}$, which has a limit, has a limit in $\bar{\Omega}$. Such sets are said to be *self-closed*. If a set $\bar{\Omega}$ is compact and self-closed it is said to be *self-compact*. The property (2) will

prove to be of fundamental importance and, for lack of a better term we venture to call a system $(\Omega; \delta^2)$ a *coherent system*.^{*} A system $(\Omega; \delta)$ is said to be *limited* if there is a positive number h such that for every $q_1 q_2$, $\delta(q_1 q_2) \leq h$.

We may have two systems $(\Omega; \delta)$ and $(\Omega; \bar{\delta})$, the set Ω being the same in each case but the two distance functions, δ and $\bar{\delta}$, not the same. If we wish to indicate that q is a limit of the sequence $\{q_n\}$ we write, in the first case, $L_n q_n = q$, and in the second case, $\bar{L}_n q_n = q$. Two systems $(\Omega; \delta)$ and $(\Omega; \bar{\delta})$ are said to be L -equivalent in case $L_n q_n = q$ implies $\bar{L}_n q_n = q$ and conversely.

Denote by \mathfrak{P} a class of elements p . If \mathfrak{P} is a subclass (reduction[†]) of Ω of a system $(\Omega; \delta)$, we denote by $(\mathfrak{P}; \delta)$ a *reduced system* where $\delta(p_1 p_2) = \delta(q_1 q_2)$ in case $p_1 = q_1$ and $p_2 = q_2$. It will be noted that, if a system $(\Omega; \delta)$ has any one of the properties (1) \cdots (5), then any reduction of $(\Omega; \delta)$ has that property also.

We speak of extremal systems $(\Omega; \delta)$, closed systems $(\Omega; \delta)$, etc., in an obvious way. A compact system is a reduction of an extremal system.

Two elements \bar{q}, \bar{q} of a set Ω of a system $(\Omega; \delta)$ are said to be connected[‡] by δ if for every e there is a finite sequence $q_{1e}, q_{2e}, \dots, q_{k_e e}$ such that $q_{1e} = \bar{q}$, $q_{k_e e} = \bar{q}$ and such that $\delta(q_{ie} q_{i+1e}) \leq e$, $i = 1, 2, \dots, k_e - 1$. The set Ω is a *connected set*, or the system $(\Omega; \delta)$ is a *connected system*, in case every pair $q_1 q_2$ of elements of Ω is connected by δ . Two classes Ω_1, Ω_2 of Ω are *directly connected* if for every e there is a q_{1e} of Ω_1 and a q_{2e} of Ω_2 such that $\delta(q_{1e} q_{2e}) \leq e$. If Ω_2 is a single element q_2 and Ω_1, Ω_2 are directly connected, then q_2 must be an element of Ω_1 or else a limit of a sequence of elements (not necessarily distinct) of Ω_1 . If two elements q_1, q_2 are directly connected then $\delta(q_1 q_2) = 0$.

2. SYSTEMS $(\Omega; \delta)$

THEOREM 1. *For every coherent system $(\Omega; \delta)$ there is an L -equivalent coherent system $(\Omega; \bar{\delta}^3)$.§*

In proof of this theorem $\bar{\delta}(q_1 q_2)$ is defined to be \underline{d} , the greatest lower bound of all d such that there is an element r_d of Ω such that $\delta(q_1 r_d) \leq d$

^{*} The importance of coherent systems is apparent from the fact that in such systems the derived class of every class is closed, cf. E. R. Hedrick, these *Transactions*, vol. 12 (1911), p. 285.

[†] Cf. E. H. Moore, *Introduction to a Form of General Analysis*, § 52. New Haven Mathematical Colloquium, Yale University Press, 1910.

[‡] Cf. *On the connection of an abstract set, etc.*, by A. D. Pitcher, *American Journal of Mathematics*, vol. 36 (1914), pp. 261-266.

§ An important consequence of this theorem is: In a coherent system two sequences which have a common limit have all limits in common.

and $\delta(r_d q_2) \leq d$. It is at once evident that $\bar{\delta}(qq) = 0$ and that

$$\bar{\delta}(q_1 q_2) = \bar{\delta}(q_2 q_1).$$

$\delta(q_1 q_2) \leq e$ implies $\bar{\delta}(q_1 q_2) \leq e$, since r_e may be taken as q_1 or q_2 . Therefore $\underline{L}_n q_n = q$ implies $\underline{L}_n \bar{q}_n = q$.

We will now show that $\underline{L}_n q_n = q$ implies $\underline{L}_n q_n = q$. It is convenient for this purpose and for the sequel to prove the following lemma.

LEMMA 1. *If $\underline{L}_n \bar{\delta}(q_{1n} q_{2n}) = 0$ there is a sequence $\{r_n\}$ of elements r_n , where for any given n , r_n may be identical with q_{1n} or q_{2n} , such that*

$$\underline{L}_n \delta(q_{1n} r_n) = 0 \quad \text{and} \quad \underline{L}_n \delta(r_n q_{2n}) = 0.$$

For every e there is an n_e such that $n \geq n_e$ implies $\bar{\delta}(q_{1n} q_{2n}) \leq e$. From the definition of $\bar{\delta}$ there is for every $n \geq n_e$ an element r_{ne} such that

$$\delta(q_{1n} r_{ne}) \leq e + \frac{1}{n} \quad \text{and} \quad \delta(r_{ne} q_{2n}) \leq e + \frac{1}{n}.$$

Let e take on a sequence of values $\{e_k\}$ such that $\underline{L}_k e_k = 0$ and consider the sequence $\{n_{e_k}\}$ which may be taken so that $n_{e_1} < n_{e_2} < \dots < n_{e_k} < n_{e_{k+1}} \dots$. Now if $n < n_{e_1}$, r_n may be taken at random. If $n_k \leq n < n_{k+1}$, take r_n to be $r_{n_{e_k}}$. It is clear that such a sequence $\{r_n\}$ satisfies the given conditions.

In case $\underline{L}_n q_n = q$ we have $\underline{L}_n \bar{\delta}(q_n q) = 0$, and therefore by the above lemma there is a sequence $\{r_n\}$ such that $\underline{L}_n \delta(q_n r_n) = 0$ and $\underline{L}_n \delta(r_n q) = 0$. Therefore $\underline{L}_n r_n = q$, and since $(\mathfrak{Q}; \delta)$ is coherent, $\underline{L}_n q_n = q$. Therefore $\underline{L}_n q_n = q$ implies $\underline{L}_n q_n = q$.

The system $(\mathfrak{Q}; \bar{\delta})$ is coherent. For suppose

$$\underline{L}_n q_{1n} = q \quad \text{and} \quad \underline{L}_n \bar{\delta}(q_{1n} q_{2n}) = 0.$$

Since $(\mathfrak{Q}; \delta)$ and $(\mathfrak{Q}; \bar{\delta})$ are \underline{L} -equivalent, $\underline{L}_n q_{1n} = q$. By the lemma there is a sequence $\{r_n\}$ such that $\underline{L}_n \delta(q_{1n} r_n) = 0$ and $\underline{L}_n \delta(r_n q_{2n}) = 0$. Since $(\mathfrak{Q}; \delta)$ is coherent $\underline{L}_n r_n = q$ and $\underline{L}_n q_{2n} = q$. But $(\mathfrak{Q}; \delta)$ and $(\mathfrak{Q}; \bar{\delta})$ are \underline{L} -equivalent and thus $\underline{L}_n q_{2n} = q$.

The system $(\mathfrak{Q}; \bar{\delta})$ has the property (3). For suppose $\underline{L}_n q_{1n} = q = \underline{L}_n q_{2n}$. Since $(\mathfrak{Q}; \delta)$ and $(\mathfrak{Q}; \bar{\delta})$ are \underline{L} -equivalent $\underline{L}_n q_{1n} = q = \underline{L}_n q_{2n}$. Then for every e there is an n_e such that $n \geq n_e$ implies $\delta(q_{1n} q) \leq e$ and $\delta(q_{2n} q) \leq e$. By definition $\bar{\delta}(q_{1n} q_{2n}) \leq e$. Therefore $\underline{L}_n \bar{\delta}(q_{1n} q_{2n}) = 0$.

THEOREM 2. *Every limited system $(\mathfrak{Q}; \delta^4)$ is a system $(\mathfrak{Q}; \delta^{45})$.*

We wish to show that there is a $\phi(e)$ such that

$$(a) \quad \underline{L}_{e \neq 0} \phi(e) = 0, \quad (b) \quad \delta(q_1 q_2) \leq e, \quad \delta(q_2 q_3) \leq e$$

implies $\delta(q_1 q_3) \leq \phi(e)$. Given e , consider all possible $q_1 q_2 q_3$ such that

$\delta(q_1 q_2) \leq e$, $\delta(q_2 q_3) \leq e$. Denote by $\phi(e)$ the least upper bound of all $\delta(q_1 q_3)$. It is obvious that $\phi(e)$ exists, since $(\Omega; \delta)$ is a limited system. It is also true that $\bigwedge_{e>0} \phi(e) = 0$. Suppose this is not true. Then there is a positive number a such that for every positive number d there is an $e \leq d$ such that $\phi(e) > a$. Thus, there is a sequence $\{e_n\}$ such that $\bigwedge_n e_n = 0$ and such that, for every n , $\phi(e_n) > a$. Thus there are sequences $\{q_{1n}\}$, $\{q_{2n}\}$, $\{q_{3n}\}$ such that $\bigwedge_n \delta(q_{1n} q_{2n}) = 0$, $\bigwedge_n \delta(q_{2n} q_{3n}) = 0$ and yet for every n $\delta(q_{1n} q_{3n}) \geq a$. This contradicts the hypothesis that we have a system $(\Omega; \delta^4)$.

THEOREM 3. *For every system $(\Omega; \delta)$ there is an \mathcal{L} -equivalent limited system $(\Omega; \bar{\delta})$.*

For if

$$\bar{\delta}(q_1 q_2) = \frac{\delta(q_1 q_2)}{1 + \delta(q_1 q_2)},$$

$(\Omega; \delta)$ and $(\Omega; \bar{\delta})$ are \mathcal{L} -equivalent and $\bar{\delta}(q_1 q_2)$ is always less than unity.

As a consequence of Theorems 2 and 3 we have the following theorem.

THEOREM 4. *For every system $(\Omega; \delta^4)$ there is an \mathcal{L} -equivalent system $(\Omega; \delta^{45})$.*

THEOREM 5. *If $(\bar{\Omega}; \bar{\delta})$ is a compact system, a reduction of a coherent system $(\Omega; \delta^3)$, then $(\bar{\Omega}; \bar{\delta})$ is a coherent system $(\bar{\Omega}; \bar{\delta}^{34})$.*

$(\bar{\Omega}; \bar{\delta})$ is coherent and has the property (3) since it is a reduction of $(\Omega; \delta)$. Suppose $(\bar{\Omega}; \bar{\delta})$ does not have the property (4). Then there are sequences $\{\bar{q}_{1n}\}$, $\{\bar{q}_{2n}\}$, $\{\bar{q}_{3n}\}$ such that $\bigwedge_n \bar{\delta}(\bar{q}_{1n} \bar{q}_{2n}) = 0$, $\bigwedge_n \bar{\delta}(\bar{q}_{2n} \bar{q}_{3n}) = 0$ and yet $\bigwedge_n \bar{\delta}(\bar{q}_{1n} \bar{q}_{3n}) \neq 0$. There are, then, sequences $\{\bar{q}_{1n_k}\}$, $\{\bar{q}_{3n_k}\}$ and an e such that for every k , $\bar{\delta}(\bar{q}_{1n_k} \bar{q}_{3n_k}) > e$. Since $(\bar{\Omega}; \bar{\delta})$ is compact there is a sequence of $\{\bar{q}_{1n_k}\}$, say $\{\bar{q}_{1n_{k_l}}\}$, which has a limit q (the sequence $\{\bar{q}_{1n_{k_l}}\}$ may be merely q repeated infinitely often). Corresponding to $\{\bar{q}_{1n_{k_l}}\}$ there are sequences $\{\bar{q}_{2n_{k_l}}\}$ and $\{\bar{q}_{3n_{k_l}}\}$ such that $\bigwedge_l \bar{\delta}(\bar{q}_{1n_{k_l}} \bar{q}_{2n_{k_l}}) = 0$ and $\bigwedge_l \bar{\delta}(\bar{q}_{2n_{k_l}} \bar{q}_{3n_{k_l}}) = 0$. Therefore, since $(\Omega; \delta)$ is coherent, $q = \bigwedge_l \bar{q}_{2n_{k_l}}$ and $q = \bigwedge_l \bar{q}_{3n_{k_l}}$. Then, since $(\Omega; \delta^3)$, also $\bigwedge_l \bar{\delta}(\bar{q}_{1n_{k_l}} \bar{q}_{3n_{k_l}}) = 0$. But this contradicts the statement that, for every k , $\bar{\delta}(\bar{q}_{1n_k} \bar{q}_{3n_k}) > e$.

THEOREM 6. *If $(\Omega; \delta)$ is a coherent system there is an \mathcal{L} -equivalent, limited, coherent system $(\Omega; \bar{\delta}^3)$ such that every compact reduction $(\bar{\Omega}; \bar{\delta})$ of $(\Omega; \bar{\delta}^3)$ is a system $(\bar{\Omega}; \bar{\delta}^5)$.*

This theorem follows at once from the successive application of Theorems 1, 3, 5, 2 and the fact that a reduction of a limited system is limited. It will be noted that to every compact reduction $(\bar{\Omega}; \bar{\delta})$ of $(\Omega; \delta)$ corresponds an \mathcal{L} -equivalent compact reduction $(\bar{\Omega}; \bar{\delta})$ of $(\Omega; \bar{\delta}^3)$. An important special case of the theorem occurs when $(\bar{\Omega}; \bar{\delta})$ is itself a compact system.

THEOREM 7. *If $(\Omega; \delta^1)$ is a coherent system then there is an \mathcal{L} -equivalent,*

limited system $(\mathfrak{Q}; \bar{\delta}^{13})$ such that on every compact set $\bar{\mathfrak{Q}}$ of \mathfrak{Q} , $\bar{\delta}^{13}$ is a voisinage, i. e., the system $(\bar{\mathfrak{Q}}; \bar{\delta}^{13})$ is a system $(\bar{\mathfrak{Q}}; \bar{\delta}^{15})$.*

This theorem follows from Theorem 6 and the fact that the property (1) is undisturbed by the transformations used in establishing the above theorems.†

3. THE FUNCTIONS OF HAHN

Fréchet‡ proved that if q_0 and q_1 are two elements of the set \mathfrak{Q} of a system $(\mathfrak{Q}; \delta^{16})$ then there is a function μ , continuous on the set \mathfrak{Q} of the system $(\mathfrak{Q}; \delta)$, such that $\mu(q_0) = 0$, $\mu(q_1) = 1$ and such that $q \neq q_0$ implies $0 < \mu(q) \leq 1$. Hahn§ proved the same theorem for a system $(\mathfrak{Q}; \delta^{15})$. The following is a generalization of this theorem of Fréchet and Hahn.

THEOREM 8. *If \mathfrak{Q}_0 and \mathfrak{Q}_1 are two subsets of the set \mathfrak{Q} of a coherent system $(\mathfrak{Q}; \delta)$, which are not directly connected, then there is a function μ , continuous on the set \mathfrak{Q} of the system $(\mathfrak{Q}; \delta)$, such that: $\mu(q) = 0$ if q is of \mathfrak{Q}_0 , $\mu(q) = 1$ if q is of \mathfrak{Q}_1 , and $0 < \mu(q) \leq 1$ if q and \mathfrak{Q}_0 are not directly connected.*||

The proof follows that of Hahn, but owing to the weaker hypotheses here employed, it seems desirable to indicate our proof.

On account of Theorem 1 we may assume that our system is a system $(\mathfrak{Q}; \delta^{23})$. Since \mathfrak{Q}_0 and \mathfrak{Q}_1 are not directly connected there is an e_1 such that if q_0 is of \mathfrak{Q}_0 and q_1 of \mathfrak{Q}_1 then $\delta(q_0 q_1) \geq e_1$. Denote by \mathfrak{Q}_2 the class of all q 's such that for every q_0 , $\delta(q_0 q) \geq e_1$. \mathfrak{Q}_2 contains \mathfrak{Q}_1 . \mathfrak{Q}_2 and \mathfrak{Q}_0 have no common limiting elements. Denote by \mathfrak{P} the class of all elements which belong neither to \mathfrak{Q}_0 nor to \mathfrak{Q}_2 . Thus \mathfrak{Q} is divided into a sequence of three classes $\mathfrak{Q}_0 \mathfrak{P} \mathfrak{Q}_2$ such that only adjacent classes may have common limiting elements and such that if $\delta(q' q'') = 0$ then q' , q'' belong to the same or to adjacent classes.

A sequence $\mathfrak{Q}_0 \mathfrak{Q}_1 \dots \mathfrak{Q}_i \mathfrak{Q}_{i+1} \dots \mathfrak{Q}_m$ of classes, which is such that only adjacent classes may have limiting elements in common and such that if $\delta(q_1 q_2) = 0$ then q_1 and q_2 belong to the same or to adjacent classes, we call a Hahn sequence. Consider the class \mathfrak{Q}_i ($i \neq 0, m$). Denote by d_{0q} the greatest lower bound of $\delta(q_0 q)$ where q is of \mathfrak{Q}_i and q_0 ranges over \mathfrak{Q}_{i-1} . Denote by d_{1q} the greatest lower bound of $\delta(q_1 q)$ where q_1 ranges over \mathfrak{Q}_{i+1} .

* The following is an example of a system $(\mathfrak{Q}; \delta)$ which is of interest here:

\mathfrak{Q} = all real numbers x such that $0 \leq x \leq 1$; $\delta(x_1 x_2) = |x_1 - x_2|$ $x_1 < 1$, $x_2 < 1$;

$$\delta(01) = \delta(10) = 1;$$

$$\delta(1x) = \delta(x1) = 1 - x \quad x \geq \frac{1}{2}; \quad \delta(1x) = 1/x \quad 0 < x < \frac{1}{2}.$$

The system $(\mathfrak{Q}; \delta)$ is extremal, coherent, perfect, but not limited.

† From Theorem 7 and the work of Chittenden (loc. cit.) it follows that in compact sets coherence and *écart* are infinitesimally equivalent.

‡ Loc. cit., § 51.

§ Monatshefte für Mathematik und Physik, vol. 19 (1908), pp. 251.

|| It is obvious that $\mu(q) = 0$ if \mathfrak{Q}_0 and q are directly connected.

If $d_{0q} \leq d_{1q}$ assign q to a class Ω_{i0} . If $d_{0q} > d_{1q}$ assign q to a class Ω_{i1} . Denote by D this principle of division.

LEMMA 2. *The principle of division D transforms a Hahn sequence into a Hahn sequence.*

In proof of this we must show that in the new sequence of classes only adjacent classes may have common limiting elements. It is clear that we need only show that Ω_{i0} and Ω_{i+1} have no common limiting elements and that Ω_{i1} and Ω_{i-1} have no common limiting elements. Suppose Ω_{i0} and Ω_{i+1} have a common limiting element q . Then there is a sequence $\{q_{0n}\}$ of Ω_{i0} and a sequence $\{q_{1n}\}$ of Ω_{i+1} which have q for a limit. Then from δ^3 it follows that $L_n \delta(q_{0n} q_{1n}) = 0$ and from the way the division of Ω_i was effected there must be a sequence $\{q_n\}$ of Ω_{i-1} such that $L_n \delta(q_n q_{0n}) = 0$. Therefore $L_n q_n = q$ and Ω_{i-1} , Ω_{i+1} have a common limiting element. This however contradicts the fact that the original sequence $\Omega_0 \cdots \Omega_i \cdots \Omega_m$ is a Hahn sequence. Similarly we may prove that Ω_{i1} and Ω_{i-1} have no common limiting elements.

We must also see that if $\delta(q_1 q_2) = 0$ then q_1 and q_2 belong to the same or to adjacent classes. It is obvious that we need only see that Ω_{i0} and Ω_{i+1} can not contain elements q_{i0} and q_{i+1} respectively such that $\delta(q_{i0} q_{i+1}) = 0$ and that Ω_{i1} and Ω_{i-1} can not contain elements q_{i1} and q_{i-1} respectively such that $\delta(q_{i1} q_{i-1}) = 0$. Suppose $\delta(q_{i0} q_{i+1}) = 0$ and that q_{i0} is of Ω_{i0} and q_{i+1} of Ω_{i+1} . Then, from the way the division is effected, Ω_{i-1} must contain an element q_{i-1} such that $\delta(q_{i-1} q_{i0}) = 0$. Then, since we have a coherent system, $\delta(q_{i-1} q_{i+1}) = 0$, which again contradicts the hypothesis that the original sequence is a Hahn sequence. Similarly for Ω_{i1} and Ω_{i-1} . Thus the lemma is proved.

Now the above sequence $\Omega_0 \mathfrak{P} \Omega_2$ is a Hahn sequence and by use of the principle of division D a succession of Hahn sequences may be formed of which the first one is

$$\Omega_0 \mathfrak{P}_0 \mathfrak{P}_1 \Omega_2,$$

and the second one,

$$\Omega_0 \mathfrak{P}_{00} \mathfrak{P}_{01} \mathfrak{P}_{10} \mathfrak{P}_{11} \Omega_2,$$

and the k th one,

$$\Omega_0 \mathfrak{P}_{00} \cdots \mathfrak{P}_{00} \mathfrak{P}_{00} \cdots \mathfrak{P}_{01} \cdots \mathfrak{P}_{i_1 i_2} \cdots \mathfrak{P}_{i_k} \cdots \mathfrak{P}_{11} \cdots \mathfrak{P}_{11} \cdots \mathfrak{P}_{11} \cdots \mathfrak{P}_{11} \Omega_2.$$

It is not difficult to show that if q is in a class $\Omega_{i_1 i_2 \cdots i_k}$ and if $\{q_n\}$ is a sequence of elements, distinct or not, such that $L_n q_n = q$ then there is an n_k and a class adjacent to $\Omega_{i_1 i_2 \cdots i_k}$ such that for $n \geq n_k$, q_n belongs to $\Omega_{i_1 i_2 \cdots i_k}$ or to this adjacent class.* Otherwise the fact that at every stage the sequence of classes is a Hahn sequence is contradicted.

Thus to each element q not of Ω_0 nor of Ω_2 is associated an infinite sequence

* It is understood here that the class $\Omega_{i_1 i_2 \cdots i_k}$ may be the class Ω_0 or Ω_2 .

of numbers,

$$i_1, i_2, i_3, \dots, i_k, \dots \quad (i_k = 0 \text{ or } 1),$$

the first k of which are identical with the subscript to the class of stage k to which q belongs. Define a function μ_1 as follows: $\mu_1(q) = 0$ if q is of \mathfrak{Q}_0 ; $\mu_1(q) = 1$ if q is of \mathfrak{Q}_2 ;

$$\mu_1(q) = \frac{i_1}{2} + \frac{i_2}{2^2} + \frac{i_3}{2^3} + \dots + \frac{i_k}{2^k} + \dots,$$

if q is of \mathfrak{P} . That μ_1 is continuous is readily seen from the foregoing and from the fact that if q_1 and q_2 are of adjacent classes of stage k then

$$|\mu_1(q_1) - \mu_2(q_2)| \leq \frac{1}{2^{k-1}}.$$

Now consider a sequence $\{e_n\}$ of numbers e_n , of which e_1 is used above, such that for every n , $e_{n-1} < e_n < e_{n+1}$ and such that $\lim e_n = 0$. Corresponding to e_1 we have μ_1 , corresponding to e_2 we have μ_2 , etc. The function $\mu = \sum_{n=1}^{\infty} \mu_n/2^n$ is a continuous function μ having the desired properties, viz., $\mu(q) = 0$ if q is of \mathfrak{Q}_0 , $\mu(q) = 1$ if q is of \mathfrak{Q}_2 , and $0 < \mu(q) \leq 1$ if \mathfrak{Q}_0 and q are not directly connected. Obviously \mathfrak{Q}_0 and \mathfrak{Q}_1 may be classes each consisting of a single element.

THEOREM 9. *A necessary and sufficient condition that every continuous function on a class $\bar{\mathfrak{Q}}$ of a class \mathfrak{Q} of a coherent system $(\mathfrak{Q}; \delta)$ be bounded and assume its bounds is that $\bar{\mathfrak{Q}}$ be self-compact.*

That the condition is sufficient is easily proved. The necessity of the condition may be proved after the manner of Hahn.* The reader who has grasped the significance of the condition (2) on δ as exemplified above will have no difficulty in carrying out the proof of Hahn under these milder hypotheses.

4. BIEXTREMAL CONNECTED SYSTEMS $(\mathfrak{Q}; \delta)$

A system $(\mathfrak{Q}; \delta)$ is said to be *biextremal* in case it is true that when two sequences $\{q_{1n}\}$, $\{q_{2n}\}$ are such that $\lim \delta(q_{1n} q_{2n}) = 0$ then there is a pair of sequences $\{q_{1n_k}\}$, $\{q_{2n_k}\}$ (subsequences of $\{q_{1n}\}$, $\{q_{2n}\}$ respectively) which have a common limit.

THEOREM 10. *Every biextremal system $(\mathfrak{Q}; \delta)$ is also extremal.*

For consider sequence $\{q_n\}$ of distinct elements. Then $\lim \delta(q_n q_n) = 0$ and there is a sequence q_{n_k} which has a limit q .

THEOREM 11. *If a function μ is continuous on a set \mathfrak{Q} of a biextremal system $(\mathfrak{Q}; \delta)$, then μ is uniformly continuous.*

For suppose μ were not uniformly continuous. Then there must be an ϵ

* Loc. cit.

such that for every n there is an element q_{1n} and an element q_{2n} such that $\delta(q_{1n} q_{2n}) < 1/n$ and yet $|\mu(q_{1n}) - \mu(q_{2n})| \geq \epsilon$. Now $\lim_n \delta(q_{1n} q_{2n}) = 0$ and since $(\Omega; \delta)$ is biextremal, there is a pair of sequences $\{q_{1n_s}\}, \{q_{2n_s}\}$ which have a common limit, say q_0 . The continuity of μ at q_0 affords a contradiction.

THEOREM 12. *If a function μ is continuous on the set Ω of a biextremal, connected system $(\Omega; \delta)$, then μ is uniformly continuous, bounded, assumes its bounds and every value between these bounds.*

This theorem follows from Theorems 10 and 11 and Fréchet,* page 8, corollary, and the paper by Pitcher already cited, page 264, Theorem III.

Thus every continuous function on a set Ω of a biextremal connected system $(\Omega; \delta)$ possesses the important properties which we usually associate with a continuous function on a closed interval or region. However in a particular case there may be very few such continuous functions. Indeed it may happen that only constant functions are continuous. For example let Ω be any set of elements and for every q_1, q_2 let $\delta(q_1 q_2) = 0$. The system thus defined is biextremal and connected. However every function continuous on the set Ω of $(\Omega; \delta)$ is constant. Certainly a proper set of continuous functions on a set Ω should, in general, contain functions other than the constant functions.

5. SYSTEMS $(\Omega; \delta)$ ADMITTING A UNIFORMLY PROPER CLASS OF CONTINUOUS FUNCTIONS

We will say that the class of all continuous functions on the set Ω of a system $(\Omega; \delta)$ is a *uniformly proper class* in case the following conditions are satisfied.

- (A) Every continuous function on Ω is bounded and assumes its bounds.
- (B) Every continuous function on Ω is uniformly continuous.
- (C) Every continuous function on Ω assumes every value between each pair of its values.
- (D) If Ω_0 and Ω_1 are two classes of Ω which are not directly connected, there is a function μ , continuous on the set Ω of the system $(\Omega; \delta)$ such that $\mu(q) = 0$ if q is of Ω_0 , $\mu(q) = 1$ if q is of Ω_1 , and $0 < \mu(q) \leq 1$ if q and Ω_0 are not directly connected.

- (E) If $q_1 \neq q_2$ there is a continuous function μ such that $\mu(q_1) \neq \mu(q_2)$.

In the sequel we give a set of conditions on $(\Omega; \delta)$ which are necessary and sufficient that the class of all continuous functions on Ω be a uniformly proper class. This set of conditions thus gives rise to the uniformly proper theory of continuous functions mentioned early in this paper.

A system $(\Omega; \delta)$ is said to be *\mathcal{L} -unique* in case no sequence $\{q_n\}$ has more than one limit q .

* Loc. cit.

THEOREM 13. *A biextremal, \mathcal{L} -unique system $(\mathfrak{Q}; \delta)$ is a coherent system.*

For suppose $q = \mathcal{L}_n q_{1n}$ and $\mathcal{L}_n \delta(q_{1n} q_{2n}) = 0$ and $q \neq \mathcal{L}_n q_{2n}$. Then there is an ϵ and a subsequence $\{q_{2n_k}\}$ of $\{q_{2n}\}$ such that for every k , $\delta(q_{2n_k} q) > \epsilon$. But $\mathcal{L}_k \delta(q_{1n_k} q_{2n_k}) = 0$ and, since $(\mathfrak{Q}; \delta)$ is biextremal, there must be sequences $\{q_{1n_{k_i}}\}$ and $\{q_{2n_{k_i}}\}$ of $\{q_{1n_k}\}$, $\{q_{2n_k}\}$ respectively which have a common limit. Since $(\mathfrak{Q}; \delta)$ is \mathcal{L} -unique this limit must be q . Thus we have a contradiction.

THEOREM 14. *The class of all continuous functions on the set \mathfrak{Q} of a biextremal, connected, \mathcal{L} -unique system $(\mathfrak{Q}; \delta)$ is a uniformly proper class.*

This theorem is a consequence of Theorems 13, 12, and 8. In applying Theorem 8 we should note that if a system $(\mathfrak{Q}; \delta)$ is \mathcal{L} -unique then $\delta(q_1 q_2) = 0$ implies $q_1 = q_2$ and no pair of classes each consisting of a single element can be directly connected.

THEOREM 15. *If a system $(\mathfrak{Q}; \delta)$ has the property E then $(\mathfrak{Q}; \delta)$ is \mathcal{L} -unique.*

For if there be a sequence with two limits q_1 and q_2 and if μ is a continuous function then $\mu(q_1) = \mu(q_2)$.

THEOREM 16. *If a system $(\mathfrak{Q}; \delta)$ has the properties A, D, E then $(\mathfrak{Q}; \delta)$ is an extremal system.*

By the previous theorem $(\mathfrak{Q}; \delta)$ is an \mathcal{L} -unique system. Suppose $(\mathfrak{Q}; \delta)$ were not an extremal system. Then there is a sequence $\{q_n\}$ of distinct elements with no limiting elements. Divide $\{q_n\}$ into two classes:

$$\mathfrak{Q}_0 = q_{j+1}, q_{j+2}, q_{j+3}, \dots, q_{j+k}, \dots; \quad \mathfrak{Q}_1 = q_1, q_2, \dots, q_j.$$

Each of these classes is closed, they are not directly connected, and, since we have \mathcal{L} -unique, neither class is directly connected with any element not in the class. Then, by the condition D, there is a continuous function μ_j such that $\mu_j(q_i) = 1$, ($i \leq j$); $\mu_j(q_i) = 0$ ($i > j$); $0 < \mu_j(q) \leq 1$ for q not of $\{q_n\}$.

Now consider the function $\mu = \sum_{j=1}^{\infty} \mu_j/2^j$, which is nowhere 0, and which is such that $\mu(q_1) = 1$, $\mu(q_2) = \frac{1}{2}$, \dots , $\mu(q_j) = 1/2^{j-1}$, \dots . Here $1/\mu$ is well defined, continuous, but not bounded. This contradicts the hypothesis that $(\mathfrak{Q}; \delta)$ is a system with the property A.

THEOREM 17. *If a system $(\mathfrak{Q}; \delta)$ has the property C then $(\mathfrak{Q}; \delta)$ is a connected system.*

This theorem is proved in the paper by Pitcher referred to above, page 265, Theorem Vb.

THEOREM 18. *If a system $(\mathfrak{Q}; \delta)$ has the properties A, B, D, E, then $(\mathfrak{Q}; \delta)$ is biextremal.*

For suppose $(\mathfrak{Q}; \delta)$ is not biextremal. Then there is a pair of sequences $\{q_{1n}\}$, $\{q_{2n}\}$ such that $\mathcal{L}_n \delta(q_{1n} q_{2n}) = 0$ but such that no sequences $\{q_{1n_k}\}$,

$\{q_{2n_k}\}$ have a common limit. We may suppose that neither $\{q_{1n}\}$ nor $\{q_{2n}\}$ contains a single element repeated infinitely often. For suppose $\{q_{1n}\}$ contains a single element q repeated infinitely often. Then this element q forms an *identical* sequence $\{q_{1n_k}\}$, $q_{1n_k} = q$ for every k . Moreover the sequences $\{q_{1n_k}\}$ and $\{q_{2n_k}\}$ have a common limit, the element q itself. This contradicts the statement that no sequences $\{q_{1n_k}\}$, $\{q_{2n_k}\}$ have a common limit. By Theorem 16, the system $(\mathfrak{Q}; \delta)$ is an extremal system. Then there is a sequence $\{q_{1n_k}\}$, a subsequence of $\{q_{1n}\}$, which has a limit q_1 . The sequence $\{q_{2n_k}\}$ contains an infinity of distinct elements and, since $(\mathfrak{Q}; \delta)$ is extremal, there is a subsequence $\{q_{2n_{k_l}}\}$ of $\{q_{2n_k}\}$ which has a limit q_2 . Thus

$$\mathcal{L}_l \delta(q_{1n_{k_l}}, q_{2n_{k_l}}) = 0, \quad \mathcal{L}_l q_{1n_{k_l}} = q_1, \quad \mathcal{L}_l q_{2n_{k_l}} = q_2.$$

By Condition *B* a function μ , continuous on \mathfrak{Q} , is uniformly continuous. Therefore $\mathcal{L}_l |\mu(q_{1n_{k_l}}) - \mu(q_{2n_{k_l}})| = 0$. Also $\mathcal{L}_l \mu(q_{1n_{k_l}}) = \mu(q_1)$ and $\mathcal{L}_l \mu(q_{2n_{k_l}}) = \mu(q_2)$. Thus if μ is continuous $\mu(q_1) = \mu(q_2)$. But this contradicts the hypothesis that $(\mathfrak{Q}; \delta)$ has the property *E*.

From Theorems 14, 15, 17, 18 we have the following theorem.

THEOREM 19. *A necessary and sufficient condition that the class of all continuous functions on the set \mathfrak{Q} of a system $(\mathfrak{Q}; \delta)$ be a uniformly proper class is that $(\mathfrak{Q}; \delta)$ be biextremal, connected, \mathcal{L} -unique.*

The significance of this theorem is emphasized by Theorem 7 and the theorem of Chittenden, previously referred to, that for every system $(\mathfrak{Q}; \delta^{15})$ there is an \mathcal{L} -equivalent system $(\mathfrak{Q}; \delta^{16})$.

6. INDEPENDENCE CONSIDERATIONS

The notion of the complete independence of a set of properties or postulates has been introduced into the literature by E. H. Moore.* We give here a set of eight examples of systems $(\mathfrak{Q}; \delta)$. Each example is preceded by a combination of plus and minus signs which indicate the character of the example as to the properties biextremal, connected, \mathcal{L} -unique. Thus $(+ - +)$ denotes that the following system $(\mathfrak{Q}; \delta)$ is biextremal, not connected, \mathcal{L} -unique.

$(+ + +) \mathfrak{Q} \equiv$ all real numbers x such that $0 \leq x \leq 1$.

$$\delta(x_1 x_2) = |x_1 - x_2|.$$

$(- + +) \mathfrak{Q} \equiv$ all real numbers x such that $0 < x \leq 1$.

$$\delta(x_1 x_2) = |x_1 - x_2|.$$

$(+ - +) \mathfrak{Q} \equiv$ all real numbers x such that $0 \leq x \leq 1$ or $2 \leq x \leq 3$.

$$\delta(x_1 x_2) = |x_1 - x_2|.$$

* Loc. cit.

$(+ + -) \mathfrak{Q} \equiv$ all real numbers x such that $0 \leq x \leq 1$ and an element $*$.

$$\delta(x_1 x_2) = |x_1 - x_2|; \quad \delta(x*) = |x - 1|.$$

$(- - +) \mathfrak{Q} \equiv$ all real numbers x such that $0 < x \leq 1$ or $2 \leq x \leq 3$.

$$\delta(x_1 x_2) = |x_1 - x_2|.$$

$(+ - -) \mathfrak{Q} \equiv$ all real numbers x such that $0 \leq x \leq 1$ or $2 \leq x \leq 3$ and an element $*$. $\delta(x_1 x_2) = |x_1 - x_2|$; $\delta(x*) = |x - 1|$.

$(- + -) \mathfrak{Q} \equiv$ all real numbers x such that $0 < x \leq 1$ and an element $*$.

$$\delta(x_1 x_2) = |x_1 - x_2|, \quad \delta(x*) = |x - 1|.$$

$(- - -) \mathfrak{Q} \equiv$ all real numbers x such that $0 < x \leq 1$ or $2 \leq x \leq 3$ and an element $*$. $\delta(x_1 x_2) = |x_1 - x_2|$; $\delta(x*) = |x - 1|$.

From these eight examples we have the following theorem.

THEOREM 20. *The properties biextremal, connected, \mathcal{L} -unique of systems $(\mathfrak{Q}; \delta)$ are completely independent.*

7. ANALYSIS OF THE PROPERTIES BIEXTREMAL AND COHERENT

We have already seen that every biextremal system $(\mathfrak{Q}; \delta)$ is extremal and that every biextremal, \mathcal{L} -unique system $(\mathfrak{Q}; \delta)$ is coherent. It is also true that every coherent, extremal system $(\mathfrak{Q}; \delta)$ is biextremal. For suppose we have a coherent, extremal system $(\mathfrak{Q}; \delta)$ and two sequences $\{q_{1n}\}, \{q_{2n}\}$ such that $\mathcal{L}_n \delta(q_{1n} q_{2n}) = 0$. We wish to show that there is a pair of sequences $\{q_{1n_k}\}, \{q_{2n_k}\}$ which have a common limit. If either $\{q_{1n}\}$ or $\{q_{2n}\}$ contain a single element repeated infinitely often our contention is established at once. If such is not the case, then, since $(\mathfrak{Q}; \delta)$ is extremal, there is a sequence $\{q_{1n_k}\}$ which has a limit q . But since $(\mathfrak{Q}; \delta)$ is coherent and

$$\mathcal{L}_k \delta(q_{1n_k} q_{2n_k}) = 0,$$

we have also $\mathcal{L}_k q_{2n_k} = q$.

THEOREM 21. *For \mathcal{L} -unique systems $(\mathfrak{Q}; \delta)$ the property biextremal is equivalent to the two properties extremal and coherent.*

Thus the four properties: extremal, connected, \mathcal{L} -unique, coherent serve equally well as the basis of a uniformly proper theory of continuous functions and in fact are both necessary and sufficient for the same. The reader can show, without serious difficulty, that these four properties are also completely independent.

The following properties are interesting and important.

(a) Every sequence $\{q_n\}$ contains a sequence $\{q_{n_k}\}$ which has a limit.

(b) If a sequence $\{q_n\}$ has a limit and $\mathcal{L}_n \delta(q_{1n} q_{2n}) = 0$, then $\{q_{1n}\}$ and $\{q_{2n}\}$ have subsequences $\{q_{1n_k}\}, \{q_{2n_k}\}$ which have a common limit.

(c) If q is a limit of a sequence $\{q_{1n}\}$ which has a limit in common with a sequence $\{q_{2n}\}$ then q is a limit of $\{q_{2n}\}$.

THEOREM 22. *The property biextremal is equivalent to properties (a) and (b) and the property coherent is equivalent to properties (b) and (c).*

The first part of the theorem is readily proved. It is also not difficult to see that every coherent system possesses the Property (b). To see that every coherent system possesses the Property (c) note that if $q_1 = \mathcal{L}_n q_{1n}$ and $\mathcal{L}_n q_{1n} = q_0 = \mathcal{L}_n q_{2n}$ then $\delta(q_1 q_0) = 0$ and $q_1 = \mathcal{L}_n q_{2n}$. Also if $(\mathfrak{Q}; \delta)$ has the Properties (b), (c) and if $\mathcal{L}_n q_{1n} = q$ and $\mathcal{L}_n \delta(q_{1n} q_{2n}) = 0$ then there is an element q_0 and sequences $\{q_{1n_k}\}, \{q_{2n_k}\}$ such that $\mathcal{L}_k q_{1n_k} = q_0 = \mathcal{L}_k q_{2n_k}$. Then from (c) q is a limit of $\{q_{2n_k}\}$. The supposition that q is not a limit of $\{q_{2n}\}$ evidently leads to a contradiction. Thus properties (b) and (c) imply the property coherent. This theorem focuses on the importance of the properties (b) and (c) and suggests sets of basic properties other than those given above.

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EXISTENCE THEOREMS FOR THE GENERAL REAL SELF-ADJOINT LINEAR SYSTEM OF THE SECOND ORDER*

BY

H. J. ETTLINGER

INTRODUCTION

Sturm† in 1836 established many fundamental theorems concerning the properties of solutions of the linear differential equation (1) below, and the system formed by (1) and the boundary conditions (3), of which the oscillation theorems of § 1 of the present paper are immediate consequences.

In the special case of periodic conditions Mason‡ and Bôcher§ with more or less generality established an oscillation theorem.

Birkhoff|| extended their work to the general self-adjoint linear boundary conditions (see (5) and (1)), where, however, he assumed $K \equiv 1$, and that λ does not enter into the boundary conditions, and established an oscillation theorem for $u_p(x)$, the solution corresponding to the p th characteristic number.

It will be the object of this paper to generalize these results to the most general real, self-adjoint linear system of the second order, where K and the coefficients of the boundary conditions are functions of λ , by extending Bôcher's and Birkhoff's methods,¶ which are based on the application of Sturm's theorems to this system.**

* Presented to the Society, under a different title, April 26, 1913.

† *Journal de mathématiques*, vol. 1 (1836), pp. 106-186.

‡ *Mathematische Annalen*, vol. 58 (1904), these *Transactions*, vol. 7 (1906), pp. 337-360.

§ *Comptes Rendus*, vol. 140 (1905), p. 928.

|| These *Transactions*, vol. 10 (1909), pp. 259-270.

¶ Some of the theorems obtained by Birkhoff have been worked out independently and by other methods by Haupt. See Dissertation, "*Ueber Oszillationstheoreme*," Teubner, Leipzig, 1911. See also *Mathematische Annalen*, vol. 76 (1914), pp. 67-104.

After a report of my results had been published (see *Bulletin of the American Mathematical Society*, vol. 19 (1913), p. 502), the case where K is a function of λ and the coefficients of the boundary conditions are constants, together with the corresponding system in difference equations was partially treated by Fort, "*Linear difference and differential equations*," *American Journal of Mathematics*, vol. 39 (1917), pp. 1-26.

** The writing of this paper has been furthered by stimulus and suggestions from Professors Birkhoff and Bôcher, to whom I desire to express my cordial appreciation.

1. ON A STURMIAN BOUNDARY PROBLEM

Given the second order linear differential equation,

$$(1) \quad \frac{d}{dx} \left[K(x, \lambda) \frac{du}{dx} \right] - G(x, \lambda)u = 0,$$

let us consider the following linear combinations of a function $u(x, \lambda)$ and its first derivative with regard to x , $u'(x, \lambda)$,

$$(2) \quad \begin{aligned} L_i[u(x, \lambda)] &= \alpha_i(\lambda)u(x, \lambda) - \beta_i(\lambda)K(x, \lambda)u'(x, \lambda) \\ M_i[u(x, \lambda)] &= \gamma_i(\lambda)u(x, \lambda) + \delta_i(\lambda)K(x, \lambda)u'(x, \lambda) \end{aligned} \quad (i = 0, 1),$$

By the sets of conditions (A), (B), for the equation (1) and the linear combinations (2) shall be meant the following.

Conditions (A).

I. $K(x, \lambda)$ and $G(x, \lambda)$ are continuous* real functions of x and λ , for all real values of x in the interval

$$(X) \quad (a \leq x \leq b)$$

and for all real values of λ in the interval

$$(\Lambda) \quad (\mathcal{L}_1 < \lambda < \mathcal{L}_2). \dagger$$

II. $K(x, \lambda)$ is always positive in (X, Λ) .

III. Throughout (Λ) , the eight coefficients $\alpha_i, \dots, \delta_i$ of (2) are continuous real functions of λ , and

$$|\alpha_i| + |\beta_i| > 0, \quad |\gamma_i| + |\delta_i| > 0.$$

IV. (i) K and G always decrease (or at least do not increase) as λ increases.

(ii) Either $\beta_i \equiv 0$, or $\beta_i \neq 0$, in which case the quotient α_i/β_i decreases (or at least does not increase) as λ increases. Also either $\delta_i \equiv 0$, or $\delta_i \neq 0$, in which case the quotient γ_i/δ_i decreases (or at least does not increase) as λ increases.

V. For any arbitrary λ , there shall exist an x for which K or G actually decreases as λ increases, unless for this value $\beta_i \neq 0$ (or $\delta_i \neq 0$) and α_i/β_i (or γ_i/δ_i) actually decreases for this value of λ as λ increases.

Conditions (B).

Such further conditions on the coefficients of the system as will ensure the correctness of Sturm's Theorem of Oscillation for the system (1), (3). Various sets of conditions have been worked out by Professor Bôcher in Chapter III,

* The existence and continuity of $\partial K / \partial x$ is commonly required, but this is not necessary. See Weyl, *Mathematische Annalen*, vol. 68 (1910), p. 221 and Bôcher, these *Transactions*, vol. 14 (1913), pp. 412-418.

† In particular, \mathcal{L}_1 and \mathcal{L}_2 may be $-\infty$ or $+\infty$ respectively.

paragraphs 13-15 of his recent book, *Leçons sur les Méthodes de Sturm* (Paris, 1917).

Consider now the sturmian system consisting of the equation (1) and conditions

$$(3) \quad \begin{aligned} L_0[u(a)] &= 0, \\ M_0[u(a)] &= 0. \end{aligned}$$

If $u(x, \lambda)$ is a solution of (1), (different from zero), satisfying the first boundary condition of (3) and if we have either

$$\frac{\gamma_0}{\delta_0} < \frac{\gamma_1}{\delta_1} \quad \text{or} \quad \frac{\gamma_0}{\delta_0} > \frac{\gamma_1}{\delta_1},$$

if $\gamma_0 \delta_1 - \delta_0 \gamma_1$ is not zero in (Λ) , and if the expressions involved satisfy (A), then it follows from Sturm's Oscillation Theorem that

$$M_0[u(b)] \quad \text{and} \quad M_1[u(b)]$$

change sign when they vanish and their roots separate one another, and also that the roots of $u(x, \lambda)$ decrease as λ increases.*

Concerning the sturmian system (1), (3) we have the following results,† provided (A) and (B) are satisfied:

I. There exist an infinite number of characteristic values‡ for the system in (Λ) , having a single cluster point at \mathcal{L}_2 .

II. If we arrange these values $\lambda_0, \lambda_1, \dots$ in order of increasing magnitude and denote the corresponding characteristic functions by $U_0(x), U_1(x), \dots$, then $U_n(x)$ will vanish precisely n times on $a < x < b$.

III. If $u(x, \lambda)$ be the particular solution satisfying

$$L_0[u(a)] = 0$$

for which

$$u(a, \lambda) = \beta_0(\lambda), \quad K(a, \lambda)u'(a, \lambda) = \alpha_0(\lambda),$$

then for $x > a$

$$\lim_{\lambda=\lambda_1} K(x, \lambda)u'(x, \lambda) = \infty, \quad \lim_{\lambda=\lambda_1} u(x, \lambda) = \infty,$$

$$\lim_{\lambda=\lambda_1} \frac{K(x, \lambda)u'(x, \lambda)}{u(x, \lambda)} = \infty.$$

* Loc. cit., pp. 139, 143.

† See Bôcher, *Encyklopädie der mathematischen Wissenschaften*, II A, 7a.

‡ A value λ_i of λ is said to be a characteristic value of a system such as (1), (3), if when $\lambda = \lambda_i$, this system has a solution not identically zero. Any such solution is termed a characteristic function.

We may notice that a very important special case of a system satisfying (A) and (B) is formed by the equation

$$\frac{d}{dx} \left[k(x) \frac{du}{dx} \right] - [l(x) - \lambda g(x)] u = 0$$

together with boundary conditions in which the coefficients are constants, provided $k(x) > 0$, $g(x) > 0$. The interval (Λ) is $(-\infty, +\infty)$.

If however g changes sign, and $l \geq 0$, $\alpha_0 \beta_0 \geq 0$, $\gamma_0 \delta_0 \geq 0$, the equation may be put in the form

$$\frac{d}{dx} \left[\frac{k(x)}{|\lambda|} \frac{du}{dx} \right] - \left[\frac{l(x)}{|\lambda|} - (\operatorname{sgn} \lambda) g(x) \right] u = 0.$$

If the parameter λ be replaced by the new parameter $\bar{\lambda} = |\lambda|$ the conditions (A) and (B) are satisfied for the interval $(0, +\infty)$ for (Λ) . By this device, due to Bôcher,* the case in which $g(x)$ changes sign can be dealt with.

2. THE GENERALIZED SELF-ADJOINT BOUNDARY PROBLEM. EXISTENCE OF AN INFINITE SET OF CHARACTERISTIC VALUES

Consider the system consisting of the differential equation (1) together with the boundary conditions

$$\begin{aligned} (4) \quad L_0[u(a)] &= M_0[u(b)], \\ L_1[u(a)] &= M_1[u(b)], \end{aligned}$$

where L_i and M_i are defined by (2). Throughout this paper we shall suppose that the system (1), (4) satisfies the set of conditions (A), § 1, and that the conditions (4) do not reduce to the sturmian type for any λ , and that these conditions are *self-adjoint*

$$\alpha_0 \beta_1 - \beta_0 \alpha_1 \equiv -(\gamma_0 \delta_1 - \delta_0 \gamma_1).$$

Now the quantity $\alpha_0 \beta_1 - \beta_0 \alpha_1$ is never zero in this case, since the conditions (4) are then reducible by linear combination to the sturmian type. By division of the first equation (4) by this quantity, we may, without loss of generality or effect upon (A), (B), § 1, take

$$(5) \quad \alpha_0 \beta_1 - \beta_0 \alpha_1 \equiv -(\gamma_0 \delta_1 - \delta_0 \gamma_1) \equiv -1.$$

We shall consider only real solutions of (1), (4).

Let $u_0(x, \lambda)$ and $u_1(x, \lambda)$ denote the two linearly independent solutions of (2) satisfying the conditions

* Cf. Bôcher, *Proceedings of The Fifth International Congress of Mathematicians*, Cambridge, England, vol. 1 (1912), p. 173.

$$(6) \quad \begin{aligned} L_0[u_0(a)] &= 0, & L_1[u_0(a)] &= 1, \\ L_0[u_1(a)] &= 1, & L_1[u_1(a)] &= 0. \end{aligned}$$

Solving the first two equations, we have

$$(7) \quad K(a, \lambda) u'_0(a, \lambda) = \alpha_0(\lambda), \quad u_0(a, \lambda) = \beta_0(\lambda);$$

and solving the second pair,

$$(8) \quad K(a, \lambda) u'_1(a, \lambda) = -\alpha_1(\lambda), \quad u_1(a, \lambda) = -\beta_1(\lambda).$$

By (A), α_0 and β_0 may not vanish together, so that (7) and (8) also determine u_0 and u_1 as functions of x , linearly independent for every value of λ . Abel's formula for (1) gives

$$(9) \quad u_0 u'_1 - u_1 u'_0 = -\frac{1}{K}.$$

Also direct computation and simplification by (6) and (9) show that

$$(10) \quad \begin{aligned} L_0[u_0] L_1[u_1] - L_0[u_1] L_1[u_0] &= -1, \\ M_0[u_0] M_1[u_1] - M_0[u_1] M_1[u_0] &= -1. \end{aligned}$$

THEOREM. *A necessary and sufficient condition that there exist a solution u not identically zero satisfying (4) when $\lambda = l$ is that $\phi(l) = 0$, where*

$$(11) \quad \phi(\lambda) = -2 + M_1[u_0(b, \lambda)] + M_0[u_1(b, \lambda)].$$

Proof. The general solution of (1) is

$$u(x, \lambda) = c_0(\lambda) u_0(x, \lambda) + c_1(\lambda) u_1(x, \lambda).$$

A simple reduction by means of (6) shows that a necessary and sufficient condition that $u(x, \lambda)$ not identically zero satisfy (4) is that c_0 and c_1 satisfy

$$\begin{aligned} c_0\{-M_0[u_0(b)]\} + c_1\{1 - M_0[u_1(b)]\} &= 0, \\ c_0\{1 - M_1[u_0(b)]\} + c_1\{-M_1[u_1(b)]\} &= 0, \end{aligned}$$

and are not both zero; this is possible if and only if

$$(12) \quad \phi(\lambda) = \begin{vmatrix} -M_0[u_0(b)] & 1 - M_0[u_1(b)] \\ 1 - M_1[u_0(b)] & -M_1[u_1(b)] \end{vmatrix} = 0.$$

By (10) this reduces to (11).

The equation $\phi(\lambda) = 0$ is the *characteristic equation*, that is, an equation which has for its roots all the characteristic values and no other roots. We may note that the quantities c_0 and c_1 are determined uniquely, except for a constant multiplier, provided not all the elements of the determinant (12) vanish. A value, l , for which $\phi(l) = 0$ but not all the elements of (12) vanish, is said to be a *simple characteristic value*.

If all the elements of (12) are zero, then c_0 and c_1 are both arbitrary and there will exist two linearly independent solutions of the system (1), (4). A value $\lambda = l$, such that

$$(13) \quad \begin{aligned} M_0[u_0(b)] &= 0, & M_0[u_1(b)] &= 1, \\ M_1[u_0(b)] &= 1, & M_1[u_1(b)] &= 0, \end{aligned}$$

is called a *double characteristic value*.*

Consider the auxiliary sturmian system

$$(14) \quad \begin{aligned} L_0[u(a)] &= 0, \\ M_0[u(b)] &= 0. \end{aligned}$$

We shall assume that this system satisfies conditions (B) of § 1 in addition to (A). Let the characteristic values of this system be $\lambda_0, \lambda_1, \dots$, ordered so that

$$\lambda_0 < \lambda_1 < \lambda_2 \dots$$

EXISTENCE THEOREM. *If the system (1), (4) satisfies conditions (A), (B), there exists at least one characteristic value of the system (1), (4) between any two characteristic values of the auxiliary sturmian system (14).*

To prove this we first note that we have already determined u_0 so as to satisfy the first of conditions (14), and the only solution which can be so determined differs from u_0 merely by a constant factor. Consequently, the characteristic values of the system (14) are the roots of the equation

$$(15) \quad M_0[u_0(b, \lambda)] = 0.$$

Hence we have from (10) for $\lambda = \lambda_i$,

$$M_0[u_1(b, \lambda_i)] M_1[u_0(b, \lambda_i)] = 1 \quad (i = 0, 1, 2, \dots).$$

Reducing (11) by means of this last relation, we have

$$(16) \quad \phi(\lambda_i) = \frac{1}{M_1[u_0(b, \lambda_i)]} \{1 - M_1[u_0(b, \lambda_i)]\}^2$$

so that at $\lambda = \lambda_i$, $\phi(\lambda_i)$ has the same sign as $M_1[u_0(b, \lambda_i)]$ except when $M_1[u_0(b, \lambda_i)] = 1$, that is when $\phi(\lambda_i) = 0$.

We may apply the results of Sturm to the functions $M_0[u_0(b, \lambda)]$ and $M_1[u_1(b, \lambda)]$, so that these functions must change sign when they vanish and their roots separate one another.† Equation (15) taken in conjunction

* Hence at a double characteristic value we have by solving (13),

$$\begin{aligned} K(b, l) u'_0(b, l) &= \gamma_0, & u_0(b, l) &= -\delta_0, \\ K(b, l) u'_1(b, l) &= -\gamma_1, & u_1(b, l) &= \delta_1. \end{aligned}$$

† See § 1.

with this fact establishes that if $M_1[u_0(b, \lambda)]$ is positive at $\lambda_0, \lambda_2, \dots$, it is negative at $\lambda_1, \lambda_3, \dots$ or *vice versa*. Hence $\phi(\lambda)$ is positive or zero at $\lambda_0, \lambda_2, \dots$, and negative at $\lambda_1, \lambda_3, \dots$; or else $\phi(\lambda)$ is positive or zero at $\lambda_1, \lambda_3, \dots$, and negative at $\lambda_0, \lambda_2, \dots$. In either case $\phi(\lambda)$ has at least one root between λ_p and λ_{p+1} . This proves the theorem and carries with it the existence of an infinite set of characteristic values for the system (1), (4) under the conditions stated.

3. SOME PROPERTIES OF THE FUNCTION $\phi(\lambda)$

In considering the uniqueness of the characteristic values, we shall need information concerning the behavior of $\phi(\lambda)$ near simple and double characteristic values.

We now introduce the further condition

$$(C) \quad \begin{vmatrix} \alpha_0 & \beta_0 & \gamma_0 & \delta_0 \\ \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \\ \alpha'_0 & \beta'_0 & \gamma'_0 & \delta'_0 \\ \alpha'_1 & \beta'_1 & \gamma'_1 & \delta'_1 \end{vmatrix} \leq 0.$$

This condition is unaltered by linear combination of the two boundary conditions.

THEOREM I'. If (A), (B), (C) are satisfied in the vicinity of $\lambda = l$, a simple characteristic value, $\phi(\lambda)$ changes sign in such wise that $d\phi/d\lambda$ has the same sign as $M_0[u_0(b, \lambda)]$ or $-M_1[u_1(b, \lambda)]$, provided that $K, G, \alpha_0, \dots, \delta_1$ have continuous derivatives with respect to λ of the first order, and for $\lambda = l$ either K_λ or G_λ is negative for at least one value of x in $a < x \leq b$.

THEOREM II'. If (A), (B), (C) are satisfied in the vicinity of $\lambda = l$, a double characteristic value, $\phi(\lambda)$ preserves a negative sign, while $M_0[u_0(b, \lambda)]$ and $-M_1[u_1(b, \lambda)]$ change from positive to negative as λ increases through l , provided that $K, G, \alpha_0, \dots, \delta_1$ have continuous derivatives with regard to λ of the first two orders, and for $\lambda = l$ either K_λ or G_λ is negative for at least one value of x in $a < x \leq b$.

Theorem I' will be proved if it is shown that $d\phi/d\lambda$ has the same sign as $M_0[u_0(b, \lambda)]$ or $-M_1[u_1(b, \lambda)]$ at $\lambda = l$. We proceed to compute $d\phi/d\lambda$. For this purpose, it will be convenient to introduce the following system of equations:

$$(17) \quad \begin{aligned} z' &= G(x, \lambda)u, \\ u' &= H(x, \mu)z, \end{aligned}$$

$$\alpha_i(\lambda)u(a) - \beta_i(\mu)z(a) = \gamma_i(\lambda)u(b) + \delta_i(\mu)z(b) \quad (i = 0, 1),$$

where

$$H(x, \mu) = 1/K(x, \mu).$$

If we eliminate z from the equations (17) we obtain

$$(17') \quad \frac{d}{dx} \left[K(x, \mu) \frac{du}{dx} \right] - G(x, \lambda) u = 0,$$

$$L_i[u(a)] = M_i[u(b)] \quad (i = 0, 1),$$

where by L_0, L_1, M_0, M_1 are denoted the same expressions as in (2) save that $\beta_0(\lambda), \beta_1(\lambda), \delta_0(\lambda), \delta_1(\lambda)$ are replaced by $\beta_0(\mu), \beta_1(\mu), \delta_0(\mu), \delta_1(\mu)$ respectively. Evidently if we write $\lambda = \mu$, the equations (17') become precisely (1), (4).

If now we define $u_0(x; \lambda, \mu), u_1(x; \lambda, \mu)$ as the solutions of the first equation (17') satisfying modified conditions like (7), (8),

$$K(a, \lambda) u'_0(a; \lambda, \mu) = \alpha_0(\lambda), \quad u_0(a; \lambda, \mu) = \beta_0(\mu),$$

$$K(a, \lambda) u'_1(a; \lambda, \mu) = -\alpha_1(\lambda), \quad u_1(a; \lambda, \mu) = -\beta_1(\mu),$$

and if, as in (11), we define

$$\phi(\lambda, \mu) = M_1[u_0(b; \lambda, \mu)] + M_0[u_1(b; \lambda, \mu)] - 2,$$

then we have clearly

$$\frac{d\phi(\lambda)}{d\lambda} = \left[\frac{\partial\phi(\lambda, \mu)}{\partial\lambda} + \frac{\partial\phi(\lambda, \mu)}{\partial\mu} \right]_{\mu=\lambda},$$

in consequence of the obvious identity $\phi(\lambda) = \phi(\lambda, \lambda)$.

If, moreover, we write

$$z_0(x; \lambda, \mu) = K(x, \mu) u'_0(x; \lambda, \mu), \quad z_1(x; \lambda, \mu) = K(x, \mu) u'_1(x; \lambda, \mu),$$

it is seen that (u_0, z_0) and (u_1, z_1) may be defined as the solutions of the first two equations (17) which fulfil the further conditions

$$z_0(a; \lambda, \mu) = \alpha_0(\lambda), \quad u_0(a; \lambda, \mu) = \beta_0(\mu),$$

$$z_1(a; \lambda, \mu) = -\alpha_1(\lambda), \quad u_1(a; \lambda, \mu) = -\beta_1(\mu).$$

We may now give $\phi(\lambda, \mu)$ the form:

$$\begin{aligned} \phi(\lambda, \mu) = & \gamma_1(\lambda) u_0(b; \lambda, \mu) + \delta_1(\mu) z_0(b; \lambda, \mu) \\ & + \gamma_0(\lambda) u_1(b; \lambda, \mu) + \delta_0(\mu) z_1(b; \lambda, \mu) - 2. \end{aligned}$$

By virtue of the symmetry in u and z on the one hand, and in λ and μ on the other, in these equations and in (17), it will be necessary to compute merely $\partial\phi/\partial\lambda$ to obtain $\partial\phi/\partial\mu$. For then replacing λ by μ , G by H , α_i by $-\beta_j$, γ_i by δ_j , u_i by z_j ($i \neq j$), we shall have $\partial\phi/\partial\mu$.

It is worth noting that the equations (5) may be used in simplifying $d\phi/d\lambda$

for $\lambda = \mu$, since these remain unaltered if α_i be replaced by $-\beta_j$, γ_i by δ_j , β_i by $-\alpha_j$, and δ_i by γ_j .

Now we have

$$(18) \quad \frac{\partial \phi}{\partial \lambda} = \gamma_0 \frac{\partial u_1(b)}{\partial \lambda} + \delta_0 \frac{\partial z_1(b)}{\partial \lambda} + \gamma_1 \frac{\partial u_0(b)}{\partial \lambda} + \delta_1 \frac{\partial z_0(b)}{\partial \lambda} \\ + \gamma'_0 u_1(b) + \gamma'_1 u_0(b),$$

where the accent indicates differentiation with respect to λ , and the arguments λ, μ are omitted in $u_0, z_0, u_1, z_1, \gamma_0, \gamma_1, \delta_0, \delta_1$. If we differentiate the first two equations (17) for $u = u_0$ with regard to λ , we obtain

$$(19) \quad \frac{d}{dx} \left[K(x, \mu) \frac{\partial u_0}{\partial \lambda} \right] - G(x, \lambda) \frac{\partial u_0}{\partial \lambda} = G_\lambda(x, \lambda) u_0.*$$

We may also notice that $\partial u_0 / \partial \lambda$ satisfies the initial conditions

$$\frac{\partial u_0(a)}{\partial \lambda} = 0, \quad \frac{\partial z_0(a)}{\partial \lambda} = K(a, \mu) \frac{\partial}{\partial \lambda} u'_0(a) = \alpha'_0(\lambda).$$

The equation (19) is a linear, non-homogeneous equation of the second order. If the right hand side is replaced by zero, u_0 and u_1 are two linearly independent solutions of the resulting homogeneous equation. Lagrange's Method of Variation of Parameters yields as the general solution

$$\frac{\partial u_0(x)}{\partial \lambda} = \int_a^x \frac{u_0(x) u_1(\xi) - u_1(x) u_0(\xi)}{u_0(\xi) u'_1(\xi) - u_1(\xi) u'_0(\xi)} \frac{G_\lambda(\xi) u_0(\xi)}{K(\xi, \mu)} d\xi \\ + C_0(\lambda) u_0(x) + C_1(\lambda) u_1(x),$$

which, after simplifying by (9) and solving for C_i to satisfy the initial conditions, becomes

$$(20) \quad \left. \frac{\partial u_0(x)}{\partial \lambda} \right|_{\mu=\lambda} = - \int_a^x [u_0(x) u_1(\xi) - u_1(x) u_0(\xi)] G_\lambda(\xi) u_0(\xi) d\xi \\ - \beta_1 \alpha'_0 u_0(x) - \beta_0 \alpha'_0 u_1(x).$$

Also by differentiating as to x and multiplying by $K(x, \mu)$, we find

$$(21) \quad \left. \frac{\partial z_0(x)}{\partial \lambda} \right|_{\mu=\lambda} = - \int_a^x [z_0(x) u_1(\xi) - z_1(x) u_0(\xi)] G_\lambda(\xi) u_0(\xi) d\xi \\ - \beta_1 \alpha'_0 z_0(x) - \beta_0 \alpha'_0 z_1(x).$$

In exactly similar fashion, we determine

$$(22) \quad \left. \frac{\partial u_1(x)}{\partial \lambda} \right|_{\mu=\lambda} = - \int_a^x [u_0(x) u_1(\xi) - u_1(x) u_0(\xi)] G_\lambda(\xi) u_1(\xi) d\xi \\ + \beta_1 \alpha'_1 u_0(x) + \beta_0 \alpha'_1 u_1(x),$$

* $G_\lambda = \frac{\partial G(x, \lambda)}{\partial \lambda}$; $K_\mu = \frac{\partial K(x, \mu)}{\partial \mu}$.

$$(23) \quad \left. \frac{\partial z_1(x)}{\partial \lambda} \right]_{\mu=\lambda} = - \int_a^x [z_0(x) u_1(\xi) - z_1(x) u_0(\xi)] G_\lambda(\xi) u_1(\xi) d\xi \\ + \beta_1 \alpha'_1 z_0(x) + \beta_0 \alpha'_1 z_1(x).$$

By means of (18), (20), (21), and (22) we may write

$$\left. \frac{\partial \phi}{\partial \lambda} \right]_{\mu=\lambda} = - \int_a^b \{M_0[u_0(b)] u_1(\xi) - M_0[u_1(b)] u_0(\xi)\} G_\lambda(\xi) u_1(\xi) d\xi \\ + \gamma'_0 u_1(b) + \gamma'_1 u_0(b) \\ - \int_a^b \{M_1[u_0(b)] u_1(\xi) - M_1[u_1(b)] u_0(\xi)\} G_\lambda(\xi) u_0(\xi) d\xi \\ + \{M_0[u_0(b)] \beta_1 \alpha'_1 + M_0[u_1(b)] \beta_0 \alpha'_1 \\ - M_1[u_0(b)] \beta_1 \alpha'_0 - M_1[u_1(b)] \beta_0 \alpha'_0\}.$$

We may write out $\partial \phi / \partial \mu$ from symmetry, and combining for $\mu = \lambda$ with $\partial \phi / \partial \lambda$, we obtain

$$(24) \quad \frac{d\phi}{d\lambda} = - \int_a^b F[u_1(\xi), u_0(\xi)] G_\lambda(\xi) d\xi \\ - \int_a^b F[u'_1(\xi), u'_0(\xi)] K_\lambda(\xi) d\xi + R,$$

where

$$F[s, t] = M_0[u_0(b)] s^2 + \{M_1[u_0(b)] - M_0[u_1(b)]\} st - M_1[u_1(b)] t^2,$$

and

$$R = M_0[u_0(b)] (\alpha'_1 \beta_1 - \alpha_1 \beta'_1) + M_1[u_0(b)] (\alpha_1 \beta'_0 - \beta_1 \alpha'_0) \\ + M_0[u_1(b)] (\alpha'_1 \beta_0 - \alpha_0 \beta'_1) - M_1[u_1(b)] (\alpha'_0 \beta_0 - \alpha_0 \beta'_0) \\ + \gamma'_1 u_0(b) + \delta'_1 z_0(b) + \gamma'_0 u_1(b) + \delta'_0 z_1(b).$$

If $\alpha_0 \cdots \delta_1$ are all constants, then $R \equiv 0$, and $d\phi/d\lambda$ reduces to the sum of two integrals, each of which has an integrand whose first factor is a quadratic form. The common discriminant of these forms is

$$D = \{M_1[u_0(b)] - M_0[u_1(b)]\}^2 + 4M_0[u_0(b)] M_1[u_1(b)].$$

From (10) we have

$$M_0[u_0(b)] M_1[u_1(b)] = -1 + M_0[u_1(b)] M_1[u_0(b)]$$

or

$$D = \{M_0[u_1(b)] + M_1[u_0(b)]\}^2 - 4.$$

The discriminant breaks up into two factors, one of which is precisely $\phi(\lambda)$ by (11). Hence at $\lambda = l$ the discriminant vanishes.

Now at a simple value, not all the coefficients of the quadratic forms can

vanish (see (10) and (13)); furthermore, u_0 and u_1 are linearly independent. Hence the first factors of the integrands will vanish only at isolated points of the interval $a < x \leq b$ for $\lambda = l$. But either G_λ or K_λ is negative for at least one value of x in $a < x \leq b$, and zero when not negative. Hence the sign of $d\phi/d\lambda$ is that of $M_1[u_1(b)]$ or $-M_0[u_0(b)]$ when $\alpha_0 \cdots \delta_1$ are constants.

Now suppose that $\alpha_0 \cdots \delta_1$ are functions of λ . Let us determine under what conditions R will have the same sign as that of the two integrals of (24). A necessary condition that the last line of R be expressible in the form

$$C_0 M_0[u_0] + D_0 M_1[u_0] + D_1 M_0[u_1] + C_1 M_1[u_1]$$

is that

$$\gamma'_1 = C_0 \gamma_0 + D_0 \gamma_1,$$

$$\delta'_1 = C_0 \delta_0 + D_0 \delta_1,$$

$$\gamma'_0 = D_1 \gamma_0 + C_1 \gamma_1,$$

$$\delta'_0 = D_1 \delta_0 + C_1 \delta_1,$$

whence solving we have

$$C_0 = \gamma'_1 \delta_1 - \gamma_1 \delta'_1, \quad C_1 = \gamma_0 \delta'_0 - \gamma'_0 \delta_0,$$

$$D_0 = \gamma_0 \delta'_1 - \gamma'_1 \delta_0, \quad D_1 = \gamma'_0 \delta_1 - \gamma_1 \delta'_0.$$

Since $D_0 = -D_1$ by (5), and since the coefficients of $M_0[u_1(b)]$ and $M_1[u_0(b)]$ in the first two lines of R are equal for the same reason, we have

$$R = \mathcal{A}M_1[u_1] + \mathcal{B}\{M_0[u_1] - M_1[u_0]\} - \mathcal{C}M_0[u_0],$$

where

$$\mathcal{A} = - \left\{ \beta_0^2 \frac{d}{d\lambda} \left(\frac{\alpha_0}{\beta_0} \right) + \delta_0^2 \frac{d}{d\lambda} \left(\frac{\gamma_0}{\delta_0} \right) \right\},$$

$$\mathcal{B} = (\alpha'_0 \beta_1 - \alpha_1 \beta'_0) + (\gamma'_0 \delta_1 - \gamma_1 \delta'_0),$$

$$\mathcal{C} = - \left\{ \beta_1^2 \frac{d}{d\lambda} \left(\frac{\alpha_1}{\beta_1} \right) + \delta_1^2 \frac{d}{d\lambda} \left(\frac{\gamma_1}{\delta_1} \right) \right\}.$$

The coefficients \mathcal{A} and \mathcal{C} are positive or zero in consequence of (A), § 1.

But we have

$$M_0[u_1] + M_1[u_0] = 2$$

at $\lambda = l$. Combining with (10) we find

$$M_0[u_1] - M_1[u_0] = \pm 2 \sqrt{-M_0[u_0] M_1[u_1]}.$$

Let $u^2 = \pm M_1[u_1]$ and $v^2 = \mp M_0[u_0]$; then it is clear that

$$R = \pm [\mathcal{A}u^2 + 2\mathcal{B}uv + \mathcal{C}v^2].$$

The necessary and sufficient condition that this quadratic form be semi-definite is that

$$\mathcal{AC} - \mathcal{B}^2 \geq 0,$$

which may be written

$$(D') \begin{vmatrix} \alpha_0 \beta'_0 - \beta_0 \alpha'_0 + \gamma_0 \delta'_0 - \delta_0 \gamma'_0 & \alpha_0 \beta'_1 - \beta_0 \alpha'_1 + \gamma_0 \delta'_1 - \delta_0 \gamma'_1 \\ \alpha_1 \beta'_0 - \beta_1 \alpha'_0 + \gamma_1 \delta'_0 - \delta_1 \gamma'_0 & \alpha_1 \beta'_1 - \beta_1 \alpha'_1 + \gamma_1 \delta'_1 - \delta_1 \gamma'_1 \end{vmatrix} \geq 0,$$

since differentiation of (5) shows that the second element of the first row of this determinant is equal to \mathcal{B} .

The determinant in (D') may be written in the form

$$\begin{aligned} & \begin{vmatrix} \alpha_0 & \beta_0 \\ \alpha'_0 & \beta'_0 \end{vmatrix} + \begin{vmatrix} \gamma_0 & \delta_0 \\ \gamma'_0 & \delta'_0 \end{vmatrix} \begin{vmatrix} \alpha_0 & \beta_0 \\ \alpha'_1 & \beta'_1 \end{vmatrix} + \begin{vmatrix} \gamma_0 & \delta_0 \\ \gamma'_1 & \delta'_1 \end{vmatrix} \\ & \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha'_0 & \beta'_0 \end{vmatrix} + \begin{vmatrix} \gamma_1 & \delta_1 \\ \gamma'_0 & \delta'_0 \end{vmatrix} \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha'_1 & \beta'_1 \end{vmatrix} + \begin{vmatrix} \gamma_1 & \delta_1 \\ \gamma'_1 & \delta'_1 \end{vmatrix} \\ & = \begin{vmatrix} \alpha_0 & \beta_0 \\ \alpha'_0 & \beta'_0 \end{vmatrix} \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha'_1 & \beta'_1 \end{vmatrix} - \begin{vmatrix} \alpha_0 & \beta_0 \\ \alpha'_1 & \beta'_1 \end{vmatrix} \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha'_0 & \beta'_0 \end{vmatrix} + \begin{vmatrix} \gamma_0 & \delta_0 \\ \gamma'_0 & \delta'_0 \end{vmatrix} \begin{vmatrix} \gamma_1 & \delta_1 \\ \gamma'_1 & \delta'_1 \end{vmatrix} \\ & - \begin{vmatrix} \gamma_0 & \delta_0 \\ \gamma'_1 & \delta'_1 \end{vmatrix} \begin{vmatrix} \gamma_1 & \delta_1 \\ \gamma'_0 & \delta'_0 \end{vmatrix} + \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha'_1 & \beta'_1 \end{vmatrix} \begin{vmatrix} \gamma_0 & \delta_0 \\ \gamma'_0 & \delta'_0 \end{vmatrix} + \begin{vmatrix} \alpha_0 & \beta_0 \\ \alpha'_0 & \beta'_0 \end{vmatrix} \begin{vmatrix} \gamma_1 & \delta_1 \\ \gamma'_1 & \delta'_1 \end{vmatrix} \\ & - \begin{vmatrix} \alpha_0 & \beta_0 \\ \alpha'_1 & \beta'_1 \end{vmatrix} \begin{vmatrix} \gamma_1 & \delta_1 \\ \gamma'_0 & \delta'_0 \end{vmatrix} - \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha'_0 & \beta'_0 \end{vmatrix} \begin{vmatrix} \gamma_0 & \delta_0 \\ \gamma'_1 & \delta'_1 \end{vmatrix}. \end{aligned}$$

But by expanding by Laplace's development we have

$$\begin{aligned} & \begin{vmatrix} \alpha_0 & \beta_0 & \alpha_0 & \beta_0 \\ \alpha_1 & \beta_1 & \alpha_1 & \beta_1 \\ \alpha'_0 & \beta'_0 & \alpha'_0 & \beta'_0 \\ \alpha'_1 & \beta'_1 & \alpha'_1 & \beta'_1 \end{vmatrix} = 2 \left\{ \begin{vmatrix} \alpha_0 & \beta_0 \\ \alpha_1 & \beta_1 \end{vmatrix} \begin{vmatrix} \alpha'_0 & \beta'_0 \\ \alpha'_1 & \beta'_1 \end{vmatrix} - \begin{vmatrix} \alpha_0 & \beta_0 \\ \alpha'_0 & \beta'_0 \end{vmatrix} \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha'_1 & \beta'_1 \end{vmatrix} \right. \\ & \left. + \begin{vmatrix} \alpha_0 & \beta_0 \\ \alpha'_1 & \beta'_1 \end{vmatrix} \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha'_0 & \beta'_0 \end{vmatrix} \right\} = 0. \end{aligned}$$

But since the system is self-adjoint,

$$\begin{vmatrix} \gamma_0 & \delta_0 \\ \gamma_1 & \delta_1 \end{vmatrix} \begin{vmatrix} \alpha'_0 & \beta'_0 \\ \alpha'_1 & \beta'_1 \end{vmatrix} = \begin{vmatrix} \alpha_0 & \beta_0 \\ \alpha_1 & \beta_1 \end{vmatrix} \begin{vmatrix} \gamma'_0 & \delta'_0 \\ \gamma'_1 & \delta'_1 \end{vmatrix} - \begin{vmatrix} \alpha_0 & \beta_0 \\ \alpha'_0 & \beta'_0 \end{vmatrix} \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha'_1 & \beta'_1 \end{vmatrix}.$$

Also

$$\begin{vmatrix} \alpha_0 & \beta_0 \\ \alpha_1 & \beta_1 \end{vmatrix} \begin{vmatrix} \gamma'_0 & \delta'_0 \\ \gamma'_1 & \delta'_1 \end{vmatrix} = \begin{vmatrix} \gamma_0 & \delta_0 \\ \gamma_1 & \delta_1 \end{vmatrix} \begin{vmatrix} \alpha'_0 & \beta'_0 \\ \alpha'_1 & \beta'_1 \end{vmatrix} - \begin{vmatrix} \gamma_0 & \delta_0 \\ \gamma'_0 & \delta'_0 \end{vmatrix} \begin{vmatrix} \gamma_1 & \delta_1 \\ \gamma'_1 & \delta'_1 \end{vmatrix}.$$

Hence

$$\begin{aligned} (D') - & \begin{vmatrix} \gamma_0 & \delta_0 \\ \gamma_1 & \delta_1 \end{vmatrix} \begin{vmatrix} \alpha'_0 & \beta'_0 \\ \alpha'_1 & \beta'_1 \end{vmatrix} - \begin{vmatrix} \alpha_0 & \beta_0 \\ \alpha_1 & \beta_1 \end{vmatrix} \begin{vmatrix} \gamma'_0 & \delta'_0 \\ \gamma'_1 & \delta'_1 \end{vmatrix} + \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha'_1 & \beta'_1 \end{vmatrix} \begin{vmatrix} \gamma_0 & \delta_0 \\ \gamma'_1 & \delta'_1 \end{vmatrix} \\ & + \begin{vmatrix} \alpha_0 & \beta_0 \\ \alpha'_0 & \beta'_0 \end{vmatrix} \begin{vmatrix} \gamma_1 & \delta_1 \\ \gamma'_1 & \delta'_1 \end{vmatrix} - \begin{vmatrix} \alpha_0 & \beta_0 \\ \alpha'_1 & \beta'_1 \end{vmatrix} \begin{vmatrix} \gamma_1 & \delta_1 \\ \gamma'_0 & \delta'_0 \end{vmatrix} - \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha'_0 & \beta'_0 \end{vmatrix} \begin{vmatrix} \gamma_0 & \delta_0 \\ \gamma'_1 & \delta'_1 \end{vmatrix} \geq 0. \end{aligned}$$

But this is the Laplace development by the first two columns of the determinant

$$-\begin{vmatrix} \alpha_0 & \beta_0 & \gamma_0 & \delta_0 \\ \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \\ \alpha'_0 & \beta'_0 & \gamma'_0 & \delta'_0 \\ \alpha'_1 & \beta'_1 & \gamma'_1 & \delta'_1 \end{vmatrix},$$

which reduces (D') to (C) .

Hence if (C) is satisfied, R , and also $d\phi/d\lambda$, has the same sign as $M_1[u_1(b)]$ or $-M_0[u_0(b)]$ at $\lambda = l$. If $\lambda = l$ is a simple characteristic value, it follows that $\phi(\lambda)$ changes sign at $\lambda = l$, having for $\lambda > l$ the same sign as $M_1[u_1(b)]$ or $-M_0[u_0(b)]$, and for $\lambda < l$ the opposite sign, at least if the functions $K, G, \alpha_0, \dots, \delta_1$ have continuous derivatives with regard to λ . Thus Theorem I' is proved.

The first part of Theorem II' will be proved if we show that $d^2\phi/d\lambda^2$ is negative at a double value. The formula obtained above for $d\phi/d\lambda$ shows that $d\phi/d\lambda$ vanishes at a double value $\lambda = l$ in consequence of equations (13). Also, if we differentiate the second identity (10) twice as to λ , we obtain at such a value

$$\frac{\partial^2 M_0[u_1]}{\partial \lambda^2} + \frac{\partial^2 M_1[u_0]}{\partial \lambda^2} = \frac{\partial M_0[u_0]}{\partial \lambda} \frac{\partial M_1[u_1]}{\partial \lambda} - \frac{\partial M_0[u_1]}{\partial \lambda} \frac{\partial M_1[u_0]}{\partial \lambda}.$$

But the left-hand member of this equation is precisely $d^2\phi/d\lambda^2$ while the right-hand member can be evaluated explicitly by use of (20), (21), (22), (23). Hence at such a double value we have

$$\begin{aligned} \frac{d^2\phi}{d\lambda^2} = & - \int_a^b \int_a^b [f^2(\xi)g^2(\eta) + f_1^2(\xi)g_1^2(\eta) + f^2(\xi)g_1^2(\eta) + f_1^2(\eta)g^2(\xi) \\ & - f(\xi)g(\xi)f(\eta)g(\eta) - f_1(\xi)g_1(\xi)f_1(\eta)g_1(\eta) \\ & - 2f(\xi)g(\xi)f_1(\eta)g_1(\eta)] d\xi d\eta + \bar{R}, \end{aligned}$$

where

$$f = u_0 \sqrt{-G_\lambda}, \quad f_1 = u'_0 \sqrt{-K_\lambda},$$

$$g = u_1 \sqrt{-G_\lambda}, \quad g_1 = u'_1 \sqrt{-K_\lambda},$$

$$\begin{aligned} \bar{R} = & -(\mathcal{A}\mathcal{C} - \mathcal{B}^2) - (E_0 + F_0) \left(- \int_a^b g^2(\xi) d\xi - \int_a^b g_1^2(\xi) d\xi \right) \\ & - (E_1 + F_1) \left(- \int_a^b f^2(\xi) d\xi - \int_a^b f_1^2(\xi) d\xi \right), \end{aligned}$$

$$E_i = \alpha'_i \beta_i - \alpha_i \beta'_i, \quad F_i = \gamma'_i \delta_i - \gamma_i \delta'_i.$$

Now by (A) , § 1, the quantities $G_\lambda, K_\lambda, E_i, F_i$ are negative or zero, so that \bar{R} is negative or zero.

It remains to determine the sign of the integral terms in $d^2\phi/d\lambda^2$. By interchanging the variables of integration in the proper manner, these terms take the form

$$-\frac{1}{2} \int_a^b \int_a^b \{ [f(\xi)g(\eta) - f(\eta)g(\xi)]^2 + [f_1(\xi)g_1(\eta) - f_1(\eta)g_1(\xi)]^2 + 2[f(\xi)g_1(\eta) - f_1(\eta)g(\xi)]^2 \} d\xi d\eta,$$

which is negative, since either f and g or f_1 and g_1 are linearly independent.

It is obvious, therefore, that $d^2\phi/d\lambda^2$ is negative at a double characteristic value. Hence $\phi(\lambda)$ preserves a negative sign at a double value.

Finally, we observe that a direct computation of $dM_0[u_0(b, \lambda)]/d\lambda$ and $dM_1[u_1(b, \lambda)]/d\lambda$ at such a double value, l , using (13), (20), (21), (22), (23) yields for $\lambda = l$,

$$\frac{dM_0[u_0(b, \lambda)]}{d\lambda} = \int_a^b f^2(\xi) d\xi + \int_a^b g^2(\xi) d\xi - (E_0 + F_0),$$

$$\frac{dM_1[u_1(b, \lambda)]}{d\lambda} = - \int_a^b f_1^2(\xi) d\xi - \int_a^b g_1^2(\xi) d\xi + E_1 + F_1.$$

These expressions are positive and negative, respectively, which proves the second part of Theorem II'.

We proceed to state and prove the above theorems in a more general form, after introducing the condition

$$(C) \quad \begin{vmatrix} \alpha_0 & \beta_0 & \gamma_0 & \delta_0 \\ \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \\ \Delta\alpha_0 & \Delta\beta_0 & \Delta\gamma_0 & \Delta\delta_0 \\ \Delta\alpha_1 & \Delta\beta_1 & \Delta\gamma_1 & \Delta\delta_1 \end{vmatrix} \leq 0.$$

THEOREM I. *If (A), (B), (C) are satisfied in the vicinity of $\lambda = l$, a simple characteristic value, $\phi(\lambda)$ changes sign in such wise that*

$$\frac{\phi(l + \Delta\lambda)}{\Delta\lambda} = \frac{\Delta\phi}{\Delta\lambda}$$

has the same sign as $M_0[u_0(b, \lambda)]$ or $-M_1[u_1(b, \lambda)]$.

THEOREM II. *If (A), (B), (C) are satisfied in the vicinity of $\lambda = l$, a double characteristic value, $\phi(\lambda)$ preserves a negative sign, while $M_0[u_0(b, \lambda)]$ and $-M_1[u_1(b, \lambda)]$ change from positive to negative as λ increases through l .*

Suppose that Theorem I did not hold for the case where the derivatives fail to exist. For instance, suppose that $\phi(\lambda)$ did not have the specified sign at $\lambda = l + \Delta\lambda > l$.

Consider modified values of $K, G, \alpha_0 \cdots \delta_1$ as follows:

$$\begin{aligned}\bar{K}(x, \lambda) &= K(x, l) + \frac{(\lambda - l)}{\Delta\lambda} [K(x, l + \Delta\lambda) - K(x, l)], \\ \bar{G}(x, \lambda) &= G(x, l) + \frac{(\lambda - l)}{\Delta\lambda} [G(x, l + \Delta\lambda) - G(x, l)], \\ (25) \quad \bar{\alpha}_0(\lambda) &= \alpha_0(l) + \frac{(\lambda - l)}{\Delta\lambda} [\alpha_0(l + \Delta\lambda) - \alpha_0(l)], \\ &\dots \dots \dots \\ \bar{\delta}_1(\lambda) &= \delta_1(l) + \frac{(\lambda - l)}{\Delta\lambda} [\delta_1(l + \Delta\lambda) - \delta_1(l)].\end{aligned}$$

The system with the modified coefficients will satisfy all the conditions as to derivatives and will coincide with the given system for $\lambda = l$ and for $\lambda = l + \Delta\lambda$. Let \bar{u}_0, \bar{u}_1 denote the two linearly independent solutions of the system (2), (6), whose coefficients are defined by (25). Let $\bar{\phi}(\lambda) = 0$ be the characteristic equation for this system. Now it is clear that $\phi(l) = \bar{\phi}(l) = 0$ and $\bar{\phi}(l + \Delta\lambda) = \phi(l + \Delta\lambda)$. The above conclusion holds for $\bar{\phi}(\lambda)$, so that at $\lambda = l$, $\bar{\phi}$ changes sign and for $\lambda > l$, $\bar{\phi}(\lambda)$ has the same sign as $\bar{M}_1[\bar{u}_1(b)]$ or $-\bar{M}_0[\bar{u}_0(b)]$. By hypothesis, however, $\phi(\lambda)$ does not have this sign at $\lambda = l + \Delta\lambda$. Hence $\bar{\phi}(\lambda)$ must vanish in the interval $(l, l + \Delta\lambda)$ for $\Delta\lambda$ arbitrarily small, i. e., $\bar{\phi}(\lambda') = 0$, where $l < \lambda' < l + \Delta\lambda$. But by Theorem I, for two successive values for which $\bar{\phi}(\lambda) = 0$, $\bar{M}_0[\bar{u}_0(b)]$ and $\bar{M}_1[\bar{u}_1(b)]$ must vanish in (l, λ') :

$$\bar{M}_0[\bar{u}_0(b, \lambda_1)] = 0, \quad \bar{M}_1[\bar{u}_1(b, \lambda_2)] = 0 \quad (l < \lambda_1, \lambda_2 \leq \lambda').$$

We infer that

$$M_0[u_0(b, l)] = 0, \quad M_1[u_1(b, l)] = 0,$$

since $\bar{M}_0[\bar{u}_0(b, \lambda)]$ and $\bar{M}_1[\bar{u}_1(b, \lambda)]$ are continuous functionals of $\bar{K}, \bar{G}, \bar{\alpha}_0, \dots, \bar{\delta}_1$ and since for $\Delta\lambda = 0$, these reduce to $K, G, \alpha_0, \dots, \delta_1$ respectively.*

Combining these equations with the equation $\phi(l) = 0$ and (10), we find further that

$$M_1[u_0(b, l)] = 1, \quad M_0[u_1(b, l)] = 1,$$

i. e., $\lambda = l$ is a double characteristic value contrary to hypothesis.

For $\lambda < l$, a similar discussion may be made.

In order to prove Theorem II, consider the system with the modified coefficients (25), which satisfies the condition as to derivatives required by Theorem II'. The values of $\bar{K}, \bar{G}, \bar{\alpha}_0, \dots, \bar{\delta}_1$, coincide with those of $K, G, \alpha_0, \dots, \delta_1$ respectively at the double value $\lambda = l$ and at $\lambda = l + \Delta\lambda$.

* Cf. Bôcher, these Transactions, vol. 3 (1902), p. 208.

Let $\bar{\phi}(\lambda) = 0$ denote the corresponding characteristic equation so that

$$\phi(l) = \bar{\phi}(l), \quad \phi(l + \Delta\lambda) = \bar{\phi}(l + \Delta\lambda).$$

If $\phi(\lambda)$ does not preserve a negative sign in the vicinity of l , we may assume that $\phi(l + \Delta\lambda)$ is positive or zero. Hence $\phi(\lambda)$ must vanish for l' inasmuch as $\lambda = l$ is a double value for (25) and $d^2 \bar{\phi}/d\lambda^2$ is negative at $\lambda = l$. At the first value l' of λ ($l' > l$) for which $\bar{\phi}(\lambda)$ vanishes, it is clear that $d\bar{\phi}/d\lambda \geq 0$ since $\bar{\phi}(\lambda)$ is negative in the vicinity of l . Therefore, as we have seen, $\bar{M}_1[\bar{u}_1(b, \lambda)]$ is negative for λ greater than but nearly equal to l . Hence this function vanishes between l and l' . This leads to a contradiction for $\Delta\lambda$ sufficiently small just as in the proof of Theorem I.

Likewise if $M_0[u_0(b, \lambda)]$ or $M_1[u_1(b, \lambda)]$ have not the stated sign near a double value of λ , we consider $\bar{M}_0[\bar{u}_0(b, \lambda)]$ or $\bar{M}_1[\bar{u}_1(b, \lambda)]$ and prove that these functions vanish between l and $l + \Delta\lambda$ where $\Delta\lambda$ is arbitrarily small. Then, by allowing $\Delta\lambda$ to approach zero, we are led to a contradiction.

Thus Theorems I and II are established.

4. ON CONDITIONS SUFFICIENT FOR UNIQUENESS

We are now ready to state a theorem concerning the uniqueness of the characteristic values.

THEOREM CONCERNING UNIQUENESS. *For a system (1), (4), satisfying (A), (B), (C), there exists one and only one characteristic value between every pair of characteristic numbers of the auxiliary sturmian system (1), (3). If $\lambda_0, \lambda_1, \lambda_2, \dots$ are the ordered characteristic numbers of this system (1), (3), and l_0, l_1, l_2, \dots are the ordered characteristic numbers of the system (1), (4) (account being taken of their multiplicity), the following cases are possible:*

$$\text{I}_a: \quad \mathcal{L}_1 < \lambda_0 \leq l_0 < \lambda_1 < l_1 \leq \lambda_2 \leq l_2 < \lambda_3 < l_3 < \dots \mathcal{L}_2,$$

$$\text{I}_b: \quad \mathcal{L}_1 < l_0 \leq \lambda_0 \leq l_1 < \lambda_1 < l_2 \leq \lambda_2 \leq l_3 < \dots \mathcal{L}_2,$$

$$\text{II}_a: \quad \mathcal{L}_1 < l_0 < \lambda_0 < l_1 \leq \lambda_1 \leq l_2 < \lambda_2 < l_3 \leq \lambda_3 \leq \dots \mathcal{L}_2,$$

$$\text{II}_b: \quad \mathcal{L}_1 < \lambda_0 < l_0 \leq \lambda_1 \leq l_1 < \lambda_2 < l_2 \leq \lambda_3 \leq l_3 < \dots \mathcal{L}_2.$$

The conditions for these cases are respectively:

$$\text{I}_a: \quad M_1[u_0(b, \lambda_0)] > 0, \quad \phi(\mathcal{L}_1 + \epsilon) > 0,$$

$$\text{I}_b: \quad M_1[u_0(b, \lambda_0)] > 0, \quad \phi(\mathcal{L}_1 + \epsilon) < 0,$$

$$\text{II}_a: \quad M_1[u_0(b, \lambda_0)] < 0, \quad \phi(\mathcal{L}_1 + \epsilon) > 0,$$

$$\text{II}_b: \quad M_1[u_0(b, \lambda_0)] < 0, \quad \phi(\mathcal{L}_1 + \epsilon) < 0.$$

Proof. Case I. If $M_1[u_0(b, \lambda_0)] > 0$, then by (16), ϕ is positive or zero

at $\lambda_0, \lambda_2, \lambda_4, \dots$ and ϕ is negative at $\lambda_1, \lambda_3, \dots$. Hence there exist at least two values (l_1, l_2) in each double interval $(\lambda_{2p-1}, \lambda_{2p+1})$ such that

$$\lambda_{2p-1} < l_1 \leq \lambda_{2p} \leq l_2 < \lambda_{2p+2}.$$

If there were additional values l_i in a double interval, there would be at least two more, or four in all, since $\phi(\lambda_{2p-1})$ and $\phi(\lambda_{2p+1})$ have the same sign. If we suppose that there is no double value, then at least two simple values must fall within $(\lambda_{2p-1}, \lambda_{2p})$ or $(\lambda_{2p}, \lambda_{2p+1})$ and at two successive such values $\Delta\phi/\Delta\lambda$ has opposite signs. However, by Theorem I of § 3, $\Delta\phi/\Delta\lambda$ would have the same sign as $-M_0[u_0(b, \lambda)]$ at both of these values. This is impossible, since the sign of $M_0[u_0(b, \lambda)]$ does not change in this interval. Suppose now that a double value exists, necessarily for $\lambda = \lambda_{2p}$ by (13). Since $\phi(\lambda_{2p-1})$ and $\phi(\lambda_{2p+1})$ are both negative and $\phi(\lambda)$ is negative near λ_{2p} by Theorem II, § 3, there will be, if there are other roots in $(\lambda_{2p-1}, \lambda_{2p+1})$, two simple roots lying in one of the intervals $(\lambda_{2p-1}, \lambda_{2p})$ or $(\lambda_{2p}, \lambda_{2p+1})$. But this has already been proved impossible.

To complete the discussion of Case I, it needs only to be shown that if $\phi(\mathcal{L}_1 + \epsilon) > 0$, we are led to a single value l_0 of l in the interval $(\mathcal{L}_1, \lambda_1)$ such that $\lambda_0 \leq l_0 \leq \lambda_1$, while if $\phi(\mathcal{L}_1 + \epsilon) < 0$, we are led to two roots l_0, l_1 , such that $\mathcal{L}_1 < l_0 \leq \lambda_0 \leq l_1 < \lambda_1$. For then we have Case I_a and Case I_b respectively.

When $\phi(\mathcal{L}_1 + \epsilon) > 0$, suppose first that $\phi(\lambda_0) > 0$. There is one and only one root in (λ_0, λ_1) since $\Delta\phi/\Delta\lambda$ has one and the same sign at all such roots which are simple. And there is no root in $(\mathcal{L}_1, \lambda_0)$, for if there were one root, there would necessarily be two which is clearly impossible. Thus we have Case I_a.

The possibility $\phi(\lambda_0) = 0$ may be excluded. For in this case λ_0 would be a double value by (10), (11), and (16). Also $\phi(\lambda)$ is negative near λ_0 by Theorem II, § 3. Hence there must be a simple characteristic value for $\lambda < \lambda_0$ at which $\Delta\phi/\Delta\lambda$ is negative. Consequently by Theorem I, § 3, $M_0[u_0(b, \lambda)]$ is negative for this value. However, by Theorem II, § 3, $M_0[u_0(b, \lambda)]$ is positive near λ_0 for $\lambda < \lambda_0$, so that $M_0[u_0(b, \lambda)]$ changes sign for $\lambda < \lambda_0$, which is impossible.

When $\phi(\mathcal{L}_1 + \epsilon) < 0$, we may apply precisely the same reasoning to the interval $(\mathcal{L}_1, \lambda_1)$, as we have done to $(\lambda_{2p-1}, \lambda_{2p+1})$ to prove that there are two roots l_0, l_1 restricted as in Case I_b.

Case II. If $M_1[u_0(b, \lambda)] < 0$, then by (16), ϕ is negative at $\lambda_0, \lambda_2, \dots$ and ϕ is positive or zero at $\lambda_1, \lambda_3, \dots$. It follows exactly as in Case I, that there exist two values l', l'' , in each double interval $(\lambda_{2p}, \lambda_{2p+2})$ such that

$$\lambda_{2p} < l' \leq \lambda_{2p+1} \leq l'' < \lambda_{2p+2}.$$

If $\phi(\mathcal{L}_1 + \epsilon) < 0$, we have Case II_b, since there are no characteristic values in $(\mathcal{L}_1, \lambda_0)$. If $\phi(\mathcal{L}_1 + \epsilon) > 0$, we have only the possibility of a single characteristic value $l_0 < \lambda_0$, and Case II_a arises.

On the basis of the foregoing Uniqueness Theorem, we can at once state the following *Oscillation Theorem* in the form of a corollary.

COROLLARY. *If the system (1), (4) satisfies (A), (B), (C), the characteristic function $u_p(x)$ belonging to the p th characteristic value will vanish $p - 2, p - 1, p, p + 1$, or $p + 2$ times on $a < x < b$.*

Proof. For the system (1), (14), the auxiliary sturmian characteristic values, λ_p , are given by

$$M_0[u_0(b, \lambda)] = 0.$$

The Uniqueness Theorem states that l_p lies on the interval $(\lambda_{p-1}, \lambda_{p+1})$. Now by Sturm's Oscillation Theorem for the system (1), (14), $u_0(x, \lambda_p)$ vanishes exactly p times on $a < x < b$ and so $u_0(x, \lambda)$ will vanish $p - 1, p$, or $p + 1$ times on $a < x < b$ for $\lambda_{p-1} \leq \lambda \leq \lambda_{p+1}$, or $u_0(x, l_p)$ will have $p - 1, p$, or $p + 1$ zeros on this same interval of the x -axis. But the zeros of $u_p(x)$ and $u_0(x, l_p)$ separate one another or else coincide, so that $u_p(x)$ will have on $a < x < b$ either $p - 2, p - 1, p, p + 1$, or $p + 2$ zeros.

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ON BOUNDARY VALUE PROBLEMS IN LINEAR DIFFERENTIAL EQUATIONS IN GENERAL ANALYSIS*

BY

T. H. HILDEBRANDT

In a paper *On a theory of linear differential equations in general analysis*,[†] we considered the solution of the general linear differential equations

$$(A) \quad M_1(\eta) = D\eta - \alpha - J\alpha\eta = \dot{0},$$

$$(B) \quad M_2(\eta) = D\eta - J\alpha\eta = 0,$$

$$(C) \quad M_2(\eta) - \alpha_0 = D\eta - J\alpha\eta - \alpha_0 = 0,$$

and their adjoints

$$(A') \quad N_1(\bar{\eta}) = D\bar{\eta} + \alpha + J\bar{\eta}\alpha = 0,$$

$$(B') \quad N_2(\bar{\eta}) = D\bar{\eta} + J\bar{\eta}\alpha = 0,$$

$$(C') \quad N_2(\bar{\eta}) - \alpha_0 = D\bar{\eta} + J\bar{\eta}\alpha - \alpha_0 = 0.$$

In the case of equations (A), (B), and (C) we found[‡] that the general solution of class \mathfrak{S}' is expressible in the form

$$\eta = \kappa + J\eta_0\kappa + \eta_1,$$

in which η_1 is a particular solution of the equation in question, η_0 is a solution of equation (A) whose Fredholm determinant is not zero, and κ is any function of the class \mathfrak{R} . Similarly, the general solution of equations (A'), (B'), and (C') has the form

$$\bar{\eta} = \kappa + J\kappa\bar{\eta}_0 + \bar{\eta}_1.$$

We found further that there is a unique solution of each of these equations which satisfies an initial condition of the form

$$\eta(x_1) = \kappa_0,$$

where x_1 is any element of \mathfrak{X} , and κ_0 any function of \mathfrak{R} .

* Presented to the Society, April 6, 1917.

† These Transactions, vol. 18 (1917), pp. 73-96. This paper will be referred to as I in the sequel. With the exception of an additional condition on J in §§ 2 and 3, the postulates and properties of the classes \mathfrak{M}' , \mathfrak{M}'' , \mathfrak{S} , \mathfrak{S}' , and \mathfrak{R} , and the functions α , η , and κ , and the operators J and D in the present paper are the same as in I.

‡ Cf. I, loc. cit., pp. 84, 86, 87.

In Section 1 of this paper we consider the solutions of the equations (A), (B), and (C), and (A'), (B'), and (C'), whose values at two or more elements of \mathfrak{X} satisfy a linear relation, i. e., a linear boundary condition. Section 2 is devoted to the definition of adjoint systems of boundary conditions. Section 3 derives the usual theorems concerning the interrelations of solutions of adjoint systems. They are in the main similar to those derived in the first three sections of the paper by Bôcher on *Applications and generalizations of the concept of adjoint systems** and include them as special cases. We have not given particular instances of the theory developed in the following pages. It is an easy matter to construct some of the more important ones along the lines outlined in §§ 12-14 of I.† We might remark, however, that if we let $\mathfrak{P}' \equiv \mathfrak{P}'' \equiv [0 \leq p \leq 1]$ and $\mathfrak{M}' \equiv \mathfrak{M}'' \equiv$ the class of all continuous functions on the interval (0, 1) and the operator J be the definite integral $\int_0^1 dp$, we have below a theory of linear boundary value problems for linear integro-differential equations.

1. Boundary conditions. The boundary conditions which we consider are linear in the values of the function η at two points x_1 and x_2 of the class \mathfrak{X} . The relations are further chosen so that the substitution of the general solution of the differential equations (A), (B), or (C) reduces the determination of the function κ to the solution of a linear general integral equation of the second or Fredholm type, i. e., one to which the Fredholm-Moore‡ theory is applicable. Such boundary conditions have the form§

$$S(\eta) = c\eta(x_1) + J\sigma_1\eta(x_1) + d\eta(x_2) + J\sigma_2\eta(x_2) = 0, \\ S(\eta) = \sigma_0,$$

in which σ_1 , σ_2 , and σ_0 are functions of the class \mathfrak{K} , and c and d are constants for which $c + d \neq 0$. By dividing by $c + d$, we get equivalent conditions with $c + d = 1$. We assume in the sequel that c and d satisfy this relation. In so far as the expression $S(\eta) + \sigma_1 + \sigma_2$ is of frequent occurrence, we denote it by $S_0(\eta)$.

In a similar way, we consider relative to the adjoint differential equations (A'), (B'), and (C') boundary conditions of the form:

$$T(\hat{\eta}) = c\hat{\eta}(x_1) + J\hat{\sigma}_1(x_1)\tau_1 + d\hat{\eta}(x_2) + J\hat{\sigma}_2(x_2)\tau_2 = 0, \\ T(\hat{\eta}) = \tau_0,$$

* These Transactions, vol. 14 (1913), pp. 403-420.

† Loc. cit., pp. 87-96.

‡ Cf. E. H. Moore: Bulletin of the American Mathematical Society, vol. 18 (1912), pp. 351-361.

§ If the classes $\mathfrak{P}' = \mathfrak{P}'' = [1, 2, \dots, n]$ and $J = \Sigma_1^n$, these conditions reduce to the boundary conditions considered by Bounitzky: Journal de Mathématiques, ser. 6, vol. 5 (1909), p. 68.

where τ_1 , τ_2 , and τ_0 are functions of the class \mathfrak{E} , and $c + d = 1$. We set

$$T_0(\hat{\eta}) = T(\hat{\eta}) + \tau_1 + \tau_2.$$

The following propositions are easily obtained by making the required substitutions and rearranging the terms properly.

- (1) $S(\eta_1 + \eta_2) = S(\eta_1) + S(\eta_2),$
- (2) $S(J\eta\kappa) = JS(\eta)\kappa,$
- (3) $S(\kappa + J\eta\kappa) = \kappa + JS_0(\eta)\kappa,$
- (4) $S_0(\eta_1 + \eta_2) = S_0(\eta_1) + S(\eta_2),$
- (5) $S_0(\kappa + \eta + J\eta\kappa) = \kappa + S_0(\eta) + JS_0(\eta)\kappa,$
- (1') $T(\hat{\eta}_1 + \hat{\eta}_2) = T(\hat{\eta}_1) + T(\hat{\eta}_2),$
- (2') $T(J\kappa\hat{\eta}) = J\kappa T(\hat{\eta}),$
- (3') $T(\kappa + J\kappa\hat{\eta}) = \kappa + J\kappa T_0(\hat{\eta}),$
- (4') $T_0(\hat{\eta}_1 + \hat{\eta}_2) = T_0(\hat{\eta}_1) + T(\hat{\eta}_2),$
- (5') $T_0(\kappa + \hat{\eta} + J\kappa\hat{\eta}) = \kappa + T_0(\hat{\eta}) + J\kappa T_0(\hat{\eta}).$

These propositions immediately give rise to the following theorems relating to the systems

- (1) $\begin{cases} M_1(\eta) = D\eta - \alpha - J\alpha\eta, \\ S_0(\eta) = 0, \end{cases}$
- (2) $\begin{cases} M_2(\eta) = D\eta - J\alpha\eta = 0, \\ S(\eta) = 0, \end{cases}$
- (3) $\begin{cases} M_2(\eta) = \alpha_0, \\ S(\eta) = \sigma_0, \end{cases}$
- (1') $\begin{cases} N_1(\hat{\eta}) = D\hat{\eta} + \alpha + J\hat{\eta}\alpha, \\ T_0(\hat{\eta}) = 0, \end{cases}$
- (2') $\begin{cases} N_2(\hat{\eta}) = D\hat{\eta} + J\hat{\eta}\alpha = 0, \\ T(\hat{\eta}) = 0, \end{cases}$
- (3') $\begin{cases} N_2(\hat{\eta}) = \alpha_0, \\ T(\hat{\eta}) = \tau_0. \end{cases}$

THEOREM I. *A necessary and sufficient condition that the system (1) [(1')] has a solution is that the Fredholm determinant of $S_0(\eta_0)$ [$T_0(\hat{\eta}_0)$] be different from zero, η_0 [$\hat{\eta}_0$] being a solution of equation (A) [(A')] whose Fredholm determinant is not zero.*

For if η_0 is a particular solution of $M_1(\eta) = 0$, whose Fredholm determinant is not zero, the general solution can be written

$$\eta = \kappa + \eta_0 + J\eta_0\kappa,$$

and so κ must satisfy the equation

$$S_0(\eta) = \kappa + S_0(\eta_0) + JS_0(\eta_0)\kappa = 0.$$

This has the form of a reciprocal relation. It has a solution and one only if the Fredholm determinant of $S_0(\eta_0)$ is not zero.*

If $S^{-1}(\eta_0)$ is the reciprocal of $S_0(\eta_0)$, then we have the

COROLLARY. *If a solution η of class \mathfrak{S}' of the system (1) exists, it has the form*

$$\eta = S^{-1}(\eta_0) + \eta_0 + J\eta_0 S^{-1}(\eta_0).$$

Denoting, for convenience, by $\eta_0 [\hat{\eta}_0]$ a solution of equation (A) [(A')] whose Fredholm determinant is not zero, we get in a similar way by using Propositions (3) [(3')]:

THEOREM II. *A necessary and sufficient condition that there exist a solution of the system (2) [(2')] which is not identically zero, is that the Fredholm determinant of $S_0(\eta_0)$ [$T_0(\hat{\eta}_0)$] be zero. If μ'_1, \dots, μ'_n [μ''_1, \dots, μ''_n] are a complete set of linearly independent solutions of*

$$\mu' + JS_0(\eta_0)\mu' = 0, \quad [\mu'' + J\mu'' T_0(\hat{\eta}_0) = 0],$$

then the general solution of the system (2) [(2')] can be written

$$\eta = \sum_{m=1}^n (\mu'_m + J\eta_0 \mu'_m) \mu''_m, \quad \left[\hat{\eta} = \sum_{m=1}^n \mu'_m (\mu''_m + J\mu''_m \hat{\eta}_0) \right],$$

where μ''_m [μ'_m] are any n functions of the class \mathfrak{M}'' [\mathfrak{M}'].

In such a case, the system (2) [(2')] is said to have n -fold compatibility.

Using propositions (1) and (3) [(1') and (3')], and denoting by $\eta_1 [\hat{\eta}_1]$ a particular solution of equation (C) [(C')], we have

THEOREM III. *The system (3) [(3')] has a unique solution if the Fredholm determinant of $S_0(\eta_0)$ [$T_0(\hat{\eta}_0)$] is not zero. If this determinant is zero, then a necessary and sufficient condition for the existence of a solution of this system is that*

$$J\kappa(S(\eta_1) - \sigma_0) = 0, \quad [J(T(\hat{\eta}_1) - \tau_0)\kappa = 0],$$

for every solution κ of the homogeneous equation

$$\kappa + JS_0(\eta_0)\kappa = 0, \quad [\kappa + JT_0(\hat{\eta}_0)\kappa = 0].$$

A similar set of propositions and theorems can be derived if the boundary conditions are linear relations in the values of the solution of the differential equations at n points of $\mathfrak{X}: x_1, \dots, x_n$. For the equation (B) the conditions take the form

$$S(\eta) = \sum_{m=1}^n ((c_m \eta(x_m) + J\sigma_m \eta(x_m))) = 0, \quad S(\eta) = \sigma_0,$$

where $\sigma_1, \dots, \sigma_n$, and σ_0 are any functions of \mathfrak{R} and c_1, \dots, c_n a set of constants for which

$$c_1 + c_2 + \dots + c_n = -1.$$

* Cf. E. H. Moore: loc. cit., p. 354.

2. On adjoint systems. Before taking up the definition of adjoint systems, it will be necessary to add a further postulate on the operator J . In this and the succeeding paragraphs, we shall assume that J has the following property:

If $J\mu'_0 \mu' = 0$ for every μ' of the class \mathfrak{M}' , then $\mu'_0 \equiv 0$,
and
if $J\mu'' \mu'_0 = 0$ for every μ'' of the class \mathfrak{M}'' , then $\mu'_0 \equiv 0$.

This property corresponds to the definite property P_0 of Moore* when $\mathfrak{P}' = \mathfrak{P}'' = \mathfrak{P}$, and $\mathfrak{M}' = \mathfrak{M}'' = \mathfrak{M}$. We shall therefore call it the *definite property* P_0 . It is equivalent to the following property, which is the form in which we shall have occasion to apply it.

If $J\kappa\kappa_0 = 0$ or $J\kappa_0\kappa = 0$ for every κ of the class \mathfrak{K} , then $\kappa_0 \equiv 0$.

We observe that J has this definite property in the finite case, and also the instances (a), (b), (c) but not (d) of I.†

Consider now the system

$$\begin{aligned} M_2(\eta) &= D\eta - J\alpha\eta, \\ (1) \quad S_1(\eta) &= c_1\eta(x_1) + J\sigma_{11}\eta(x_1) + d_1\eta(x_2) + J\sigma_{12}\eta(x_2). \end{aligned}$$

We assume that the expression $S_1(\eta)$ is what we shall call *linearly independent*, i. e., there exists another linear combination of $\eta(x_1)$ and $\eta(x_2)$

$$S_2(\eta) = c_2\eta(x_1) + J\sigma_{21}\eta(x_1) + d_2\eta(x_2) + J\sigma_{22}\eta(x_2),$$

with $c_1d_2 - c_2d_1 \neq 0$, such that the equations $S_1(\eta) = 0$ and $S_2(\eta) = 0$ have the unique solution

$$\eta(x_1) = 0 \quad \text{and} \quad \eta(x_2) = 0.‡$$

We can always determine the constants a and b with $b \neq 0$ so that in

$$S'_2 = aS_1 + bS_2,$$

$$c'_2 = c_1 - 1, \quad d'_2 = 2 - c_1,$$

i. e.,

$$c'_2 + d'_2 = 1, \quad c_1d'_2 - c'_2d_1 = 1.$$

Evidently S_1 and S'_2 will be completely equivalent to S_1 and S_2 , and we shall assume in the sequel that in the S_2 chosen, c_2 and d_2 have the character of c'_2 and d'_2 .

* Loc. cit., p. 361.

† Loc. cit., pp. 89-92.

‡ If $\mathfrak{P}' = \mathfrak{P}'' = (1, \dots, n)$, and $J = \sum_{m=1}^{m=n}$ then this condition actually reduces to the linear independence of the boundary conditions S_1 . A discussion of the above definition for the case of linear integral expressions by L. J. Rouse will appear shortly.

On account of the assumption relative to S_1 and S_2 , it is possible to solve the equations

$$c_1 \eta(x_1) + J\sigma_{11} \eta(x_1) + d_1 \eta(x_2) + J\sigma_{12} \eta(x_2) = S_1,$$

$$c_2 \eta(x_1) + J\sigma_{21} \eta(x_1) + d_2 \eta(x_2) + J\sigma_{22} \eta(x_2) = S_2,$$

for $\eta(x_1)$ and $\eta(x_2)$ in terms of S_1 and S_2 . If we substitute in the expression

$$J\hat{\eta}(x_2)\eta(x_2) - J\hat{\eta}(x_1)\eta(x_1)$$

and collect the coefficients of S_1 and S_2 , these coefficients will be linear expressions in $\hat{\eta}(x_1)$ and $\hat{\eta}(x_2)$ of the form

$$c'_1 \hat{\eta}(x_1) + J\hat{\eta}(x_1)\tau_{11} + d'_1 \hat{\eta}(x_2) + J\hat{\eta}(x_2)\tau_{12} = T_1(\hat{\eta}),$$

$$c'_2 \hat{\eta}(x_1) + J\hat{\eta}(x_1)\tau_{21} + d'_2 \hat{\eta}(x_2) + J\hat{\eta}(x_2)\tau_{22} = T_2(\hat{\eta}).$$

We therefore have the identity

$$J\hat{\eta}(x_2)\eta(x_2) - J\hat{\eta}(x_1)\eta(x_1) = JT_1(\hat{\eta})S_2(\eta) - JT_2(\hat{\eta})S_1(\eta),$$

and we note that on account of the condition J^{P_0} , the $T_1(\hat{\eta})$ and $T_2(\hat{\eta})$ will be uniquely defined, if S_1 and S_2 are given as functions of $\eta(x_1)$ and $\eta(x_2)$.

We call the expression $N_2(\hat{\eta}) = D\hat{\eta} + J\hat{\eta}\alpha$ together with the $T_1(\hat{\eta})$ so defined an *adjoint system* of system (1).*

In order to obtain the relations between the constants and functions in S_1 , S_2 and T_1 , T_2 , we can proceed as outlined in the definition. The same relations result however if we equate the coefficients in the defining identity. This yields

$$c'_1 c_2 - c'_2 c_1 = -1, \quad d'_1 c_2 - d'_2 c_1 = 0,$$

$$c'_1 d_2 - c'_2 d_1 = 0, \quad d'_1 d_2 - d'_2 d_1 = 1;$$

from which we conclude that

$$c'_1 = d_1, \quad d'_1 = c_1, \quad c'_2 = d_2, \quad d'_2 = c_2.$$

We find further

$$d_1 \sigma_{21} - d_2 \sigma_{11} + c_2 \tau_{11} - c_1 \tau_{21} + J(\tau_{11} \sigma_{21} - \tau_{21} \sigma_{11}) = 0,$$

$$d_1 \sigma_{22} - d_2 \sigma_{12} + d_2 \tau_{11} - d_1 \tau_{21} + J(\tau_{11} \sigma_{22} - \tau_{21} \sigma_{12}) = 0,$$

* This definition is a generalization of the definition due to Birkhoff, these *Transactions*, vol. 9 (1908), p. 173. Bounitzky (loc. cit., p. 73) gives a definition of adjoint which may be generalized as follows: $S_1(\eta)$ and $T_1(\hat{\eta})$ are adjoint if for every $\eta(x_1)$, $\eta(x_2)$, $\hat{\eta}(x_1)$, $\hat{\eta}(x_2)$ for which $S_1(\eta) = 0$ and $T_1(\hat{\eta}) = 0$, we have

$$J\hat{\eta}(x_2)\eta(x_2) - J\hat{\eta}(x_1)\eta(x_1) = 0.$$

If S_1 and T_1 are adjoint according to our definition they will also be according to the Bounitzky definition, and conversely on account of J^{P_0} . The Bounitzky definition does not however seem to lend itself so readily to the derivation of the results of this section and the next.

$$c_1 \sigma_{21} - c_2 \sigma_{11} + c_2 \tau_{12} - c_1 \tau_{22} + J(\tau_{12} \sigma_{21} - \tau_{22} \sigma_{11}) = 0,$$

$$c_1 \sigma_{22} - c_2 \sigma_{12} + d_2 \tau_{12} - d_1 \tau_{22} + J(\tau_{12} \sigma_{22} - \tau_{22} \sigma_{12}) = 0.$$

These equalities express the fact that the system of kernels

$$c_1 \tau_{21} - c_2 \tau_{11}, \quad d_1 \tau_{21} - d_2 \tau_{11},$$

$$c_2 \tau_{12} - c_1 \tau_{22}, \quad d_2 \tau_{12} - d_1 \tau_{22},$$

are the reciprocals of the system of kernels

$$d_2 \sigma_{11} - d_1 \sigma_{21}, \quad d_2 \sigma_{12} - d_1 \sigma_{22},$$

$$c_1 \sigma_{21} - c_2 \sigma_{11}, \quad c_1 \sigma_{22} - c_2 \sigma_{12}.$$

If we add the last four equations, we get the following symmetrical relation

$$\begin{aligned} \sigma_{22} + \sigma_{21} + \tau_{11} + \tau_{12} + J(\tau_{11} + \tau_{12})(\sigma_{22} + \sigma_{21}) \\ = \sigma_{11} + \sigma_{12} + \tau_{21} + \tau_{22} + J(\tau_{21} + \tau_{22})(\sigma_{11} + \sigma_{12}). \end{aligned}$$

With the aid of these relations we verify without much difficulty that

$$\begin{aligned} T_{01}(\hat{\eta}) + S_{02}(\eta) + JT_{01}(\hat{\eta})S_{02}(\eta) - T_{02}(\hat{\eta}) - S_{01}(\eta) \\ - JT_{02}(\hat{\eta})S_{01}(\eta) = \hat{\eta}(x_2) + \eta(x_2) + J\hat{\eta}(x_2)\eta(x_2) \\ - \hat{\eta}(x_1) - \eta(x_1) - J\hat{\eta}(x_1)\eta(x_1), \end{aligned}$$

where

$$T_{01}(\hat{\eta}) = T_1(\hat{\eta}) + \tau_{11} + \tau_{12}, \quad T_{02}(\hat{\eta}) = T_2(\hat{\eta}) + \tau_{21} + \tau_{22},$$

$$S_{01}(\eta) = S_1(\eta) + \sigma_{11} + \sigma_{12}, \quad S_{02}(\eta) = S_2(\eta) + \sigma_{21} + \sigma_{22}.$$

We therefore have

THEOREM I. *If $S_1(\eta)$ and $T_1(\hat{\eta})$ are adjoint, i. e., if we have identically*

$$J\hat{\eta}(x_2)\eta(x_2) - J\hat{\eta}(x_1)\eta(x_1) = JT_1S_2 - JT_2S_1,$$

then we also have identically

$$\begin{aligned} \hat{\eta}(x_2) + \eta(x_2) + J\hat{\eta}(x_2)\eta(x_2) - \hat{\eta}(x_1) - \eta(x_1) - J\hat{\eta}(x_1)\eta(x_1) \\ = T_{01} + S_{02} + JT_{01}S_{02} - T_{02} - S_{01} - JT_{02}S_{01}. \end{aligned}$$

The same relations between the σ and τ in the light of their reciprocal character give

THEOREM II. *If T_1 and T_2 are adjoint to S_1 and S_2 then from*

$$T_1(\hat{\eta}) = 0 \quad \text{and} \quad T_2(\hat{\eta}) = 0$$

it follows uniquely that

$$\hat{\eta}(x_1) = 0 \quad \text{and} \quad \hat{\eta}(x_2) = 0,$$

i. e., $T_1(\hat{\eta})$ is also linearly independent.

The element of arbitrariness which enters into the definition of the adjoint T_1 of S_1 is taken care of in the following

THEOREM III. *If S_2 be replaced by any other S'_2 which has the same character as S_2 , and if we denote by T_1 and T'_1 the corresponding adjoint expressions, then there exist functions κ_1 and κ'_1 of the class \mathfrak{R} such that*

$$T_1 = T'_1 + JT'_1 \kappa_1$$

and

$$T'_1 = T_1 + JT_1 \kappa'_1,$$

i. e., T_1 and T'_1 are essentially equivalent.

For if we solve the relations

$$S_1 = c_1 \eta(x_1) + J\sigma_{11} \eta(x_1) + d_1 \eta(x_2) + J\sigma_{12} \eta(x_2),$$

$$S_2 = c_2 \eta(x_1) + J\sigma_{21} \eta(x_1) + d_2 \eta(x_2) + J\sigma_{22} \eta(x_2)$$

for $\eta(x_1)$ and $\eta(x_2)$, we find

$$\eta(x_1) = d_2 S_1 - d_1 S_2 + J\tau_{21} S_1 - J\tau_{11} S_2,$$

$$\eta(x_2) = c_1 S_2 - c_2 S_1 - J\tau_{22} S_1 + J\tau_{12} S_2.$$

Substituting these values in

$$S'_2 = c_2 \eta(x_1) + J\sigma'_{21} \eta(x_1) + d_2 \eta(x_2) + J\sigma'_{22} \eta(x_2),$$

we find

$$S'_2 = S_2 + J(\kappa_2 S_1 + \kappa_1 S_2),$$

where κ_1 and κ_2 are functions of the class \mathfrak{R} , depending on σ'_{21} , σ'_{22} , τ_{11} , τ_{12} , τ_{21} , and τ_{22} . We therefore have

$$\begin{aligned} J\hat{\eta}(x_2) \eta(x_2) - J\hat{\eta}(x_1) \eta(x_1) &= JT'_1 S'_2 - JT'_2 S_1 \\ &= JT'_1 (S_2 + J\kappa_2 S_1 + J\kappa_1 S_2) - JT'_2 S_1 \\ &= J(T'_1 + JT'_1 \kappa_1) S_2 - J(T'_2 - JT'_1 \kappa_2) S_1; \end{aligned}$$

from which on account of the uniqueness of T_1 it follows that

$$T_1 = T'_1 + JT'_1 \kappa_1.$$

In a similar way we obtain the other relation of the theorem.

3. On the solutions of adjoint systems. Before taking up the relations which exist between solutions of adjoint systems we note the following lemmas.

LEMMA I. *If η is a solution of equation (A) and $\hat{\eta}$ a solution of the adjoint equation (A'), then we have for every set of adjoint expressions S_1, S_2, T_1 , and T_2*

$$T_{01}(\hat{\eta}) + S_{02}(\eta) + JT_{01}(\hat{\eta})S_{02}(\eta) = T_{02}(\hat{\eta}) + S_{01}(\eta) + JT_{02}(\hat{\eta})S_{01}(\eta).$$

For by Theorem IV' of § 11 of I,* we have for every pair of solutions η and $\hat{\eta}$ of the adjoint equations (A) and (A')

$$\hat{\eta}(x_2) + \eta(x_2) + J\hat{\eta}(x_2)\eta(x_2) = \hat{\eta}(x_1) + \eta(x_1) + J\hat{\eta}(x_1)\eta(x_1).$$

Our lemma is then an immediate consequence of Theorem I of the preceding section.

LEMMA II. *If η is any solution of the system (2) [(2')]*

$$M_2(\eta) = 0, \quad S_1(\eta) = 0, \quad [N_2(\hat{\eta}) = 0, \quad T_1(\hat{\eta}) = 0],$$

then $S_2(\eta) [T_2(\hat{\eta})]$ is a solution of the equation

$$S_2(\eta) + JT_{01}(\hat{\eta})S_2(\eta) = 0, \quad [T_2(\hat{\eta}) + JT_2(\hat{\eta})S_{01}(\eta) = 0],$$

where $\hat{\eta} [\eta]$ is any solution of equation (A') [(A)].

Let η be a solution of system (2). Then if η_0 is any solution of equation (A), $\eta_0 + \eta$ is also a solution of equation (A). Further by proposition (4) of § 1, we have

$$S_{01}(\eta_0 + \eta) = S_{01}(\eta_0) + S_1(\eta) = S_{01}(\eta_0),$$

$$S_{02}(\eta_0 + \eta) = S_{02}(\eta_0) + S_2(\eta).$$

Applying Lemma I to the solutions η_0 and $\eta_0 + \eta$, we have for every solution $\hat{\eta}$ of the adjoint equation (A')

$$\begin{aligned} T_{01}(\hat{\eta}) + S_{02}(\eta_0) + S_2(\eta) + JT_{01}(\hat{\eta})S_{02}(\eta_0) + JT_{01}(\hat{\eta})S_2(\eta) \\ = T_{02}(\hat{\eta}) + S_{01}(\eta_0) + JT_{02}(\hat{\eta})S_{01}(\eta_0) \\ = T_{01}(\hat{\eta}) + S_{02}(\eta_0) + JT_{01}(\hat{\eta})S_{02}(\eta_0). \end{aligned}$$

Hence

$$S_2(\eta) + JT_{01}(\hat{\eta})S_2(\eta) = 0.$$

The proof for the lemma in the brackets runs parallel.

We are now in position to derive the following theorems.

THEOREM I. *If the system (1)*

$$M_1(\eta) = 0, \quad S_{01}(\eta) = 0$$

has a unique solution, then the adjoint system (1')

$$N_1(\hat{\eta}) = 0, \quad T_{01}(\hat{\eta}) = 0$$

also has a unique solution and conversely.

Suppose η is the unique solution of system (1), and let $\hat{\eta}_0$ be any solution of equation (A'), whose Fredholm determinant is not zero. Then by Lemma I we shall have for η and $\hat{\eta}_0$

$$S_{02}(\eta) + T_{01}(\hat{\eta}_0) + JT_{01}(\hat{\eta}_0)S_{02}(\eta) = T_{02}(\hat{\eta}_0).$$

* Loc. cit., p. 87.

If we consider this as an integral equation in $S_{02}(\eta)$, it has a unique solution, viz., the value of $S_{02}(\eta)$. Hence the Fredholm determinant of $T_{01}(\eta_0)$ is not zero, which by Theorem I of § 1 is a necessary and sufficient condition for the existence of a unique solution of the system (1').

THEOREM II. *If the system (2)*

$$M_2(\eta) = 0, \quad S_1(\eta) = 0$$

has n-fold compatibility, then the system (2')

$$N_2(\eta) = 0, \quad T_1(\eta) = 0$$

has also n-fold compatibility, and conversely.

Let μ'_1, \dots, μ'_n be a complete system of linearly independent solutions of the homogeneous equation

$$\mu' + JS_{01}(\eta_0)\mu' = 0,$$

η_0 being a solution of equation (A) whose Fredholm determinant is not zero. Then by Theorem II of § 1

$$\eta = \sum_{m=1}^n (\mu'_m + J\eta_0 \mu'_m) \mu''_m$$

μ''_m being arbitrary, is a general solution of system (2). By Lemma II

$$S_2(\eta) = \sum_{m=1}^n (\mu'_m + JS_{02}(\eta_0)\mu'_m) \mu''_m$$

will satisfy the equation

$$\kappa + JT_{01}(\eta_0)\kappa = 0,$$

i. e., we shall have

$$\sum_{m=1}^n (\mu'_m + JS_{02}(\eta)\mu'_m + JT_{01}(\eta_0)(\mu'_m + JS_{02}(\eta)\mu'_m)) \mu''_m = 0.$$

Since the μ''_m are arbitrary, we conclude that the equation

$$\mu' + JT_{01}(\eta_0)\mu' = 0,$$

η_0 being a solution of equation (A') whose Fredholm determinant is not zero, has a set of n solutions

$$\mu'_m + JS_{02}(\eta_0)\mu'_m.$$

These solutions are linearly independent. For suppose they were not. Then there would exist n constants c_1, \dots, c_n , such that

$$\sum_{m=1}^n c_m (\mu'_m + JS_{02}(\eta_0)\mu'_m) = \sum_{m=1}^n c_m \mu'_m + JS_{02}(\eta_0) \sum_{m=1}^n c_m \mu'_m = 0.$$

Let

$$\sum_{m=1}^n c_m \mu'_m = \mu'.$$

Then for every μ'' the function

$$\eta = \mu' \mu'' + J\eta_0 \mu' \mu''$$

will satisfy both of the conditions

$$S_1(\eta) = 0, \quad S_2(\eta) = 0.$$

But from our assumption relative to S_1 and S_2 it follows that

$$\eta(x_1) = 0, \quad \eta(x_2) = 0,$$

from which we have $\mu' \equiv 0$, which is contrary to our assumption that the μ'_m are linearly independent. It follows therefore that the expressions

$$\mu'_m + JS_{02}(\eta_0)\mu'_m$$

are linearly independent.

Now if the linear equation

$$\mu' + JT_{01}(\hat{\eta}_0)\mu' = 0$$

has at least n linearly independent solutions, then the adjoint linear integral equation

$$\mu'' + J\mu'' T_{01}(\hat{\eta}_0) = 0$$

will also have at least n linearly independent solutions, so that by Theorem II of § 1, we know that the system (2') has at least n -fold compatibility.

By following through a similar line of reasoning, we show that if the system (2') has m -fold compatibility, the system (2) has at least m -fold compatibility. But from this we conclude that $m = n$, which is the desired result.

Since 0-fold compatibility of system (2) yields a unique solution for the system (1), Theorem I may be regarded as a corollary to Theorem II.

We note further that in the course of the proof we have extended Lemma II to

LEMMA IIa. *Every solution κ of the equation*

$$\kappa + JT_{01}(\hat{\eta}_0)\kappa = 0, \quad [\kappa + J\kappa S_{01}(\eta_0) = 0]$$

when $\hat{\eta}_0[\eta_0]$ is a solution of equation (A') [(A)] whose Fredholm determinant is not zero, is expressible in the form $S_2(\eta)[T_2(\hat{\eta})]$, where $\eta[\hat{\eta}]$ is a solution of the system (2) [(2')].

THEOREM III. *A necessary and sufficient condition for the existence of a solution of the non-homogeneous system (3) [(3')]*

$$M_2(\eta) = \alpha_0, \quad S_1(\eta) = \sigma_0, \quad [N_2(\hat{\eta}) = \alpha_0, \quad T_1(\hat{\eta}) = \tau_0]$$

is that

$$\int_{x_1}^{x_2} J\hat{\eta}\alpha_0 = -JT_2(\hat{\eta})\sigma_0, \quad \left[\int_{x_1}^{x_2} J\alpha_0\eta = -J\tau_0 S_2(\eta) \right]$$

for every solution $\hat{\eta}[\eta]$ of the adjoint homogeneous system (2') [(2)].

Suppose η is a solution of the system (3). Then if we let $\hat{\eta}$ be any solution of the adjoint system (2') and apply Green's Theorem G_2^*

$$\begin{aligned}\int_{x_1}^{x_2} J(\hat{\eta}M_2(\eta) + N_2(\hat{\eta})\eta) &= J\hat{\eta}(x_2)\eta(x_2) - J\hat{\eta}(x_1)\eta(x_1) \\ &= JT_1(\hat{\eta})S_2(\eta) - JT_2(\hat{\eta})S_1(\eta),\end{aligned}$$

we get at once the condition of the theorem, i. e., the condition is necessary.

On the other hand, the condition is sufficient. For if η_1 be a particular solution of equation (C), and $\hat{\eta}$ be a solution of the system (2'), we get by the Green's Theorem G_2

$$\int_{x_1}^{x_2} J\hat{\eta}\alpha_0 = -JT_2(\hat{\eta})S_1(\eta_1).$$

Applying the conditions of our theorem, we have for any particular solution η_1 of equation (C) and every solution $\hat{\eta}$ of system (2')

$$JT_2(\hat{\eta})(S_1(\eta_1) - \sigma_0) = 0.$$

By Lemma IIa however, every solution of the equation

$$\kappa + J\kappa S_{01}(\eta_0) = 0$$

is expressible in the form $T_2(\hat{\eta})$, i. e., we have for every such κ

$$J\kappa(S_1(\eta_1) - \sigma_0) = 0.$$

By Theorem III of § 1, this is sufficient for the existence of a solution of system (3).

ANN ARBOR, MICH.,
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* Cf. I, loc. cit., p. 86.

A FUNDAMENTAL SYSTEM OF FORMAL COVARIANTS MODULO 2 OF THE BINARY CUBIC*

BY

OLIVER EDMUNDS GLENN

If a binary form of order m ,

$$f_m = (a_0, a_1, \dots, a_m | x_1, x_2)^m,$$

whose coefficients are arbitrary variables, be transformed by the group A of all linear substitutions on x_1, x_2 , whose coefficients are least positive residues modulo p , a prime number, there is brought into existence an infinitude of rational integral functions of $a_0, \dots, a_m, x_1, x_2$, which are invariants under the group. Whether this infinite system possesses the property of finiteness, in general, is an unsolved problem, but in this paper I show that, when the modulus is 2, the system of covariants of a cubic f_3 is finite and that the fundamental set consists of twenty quantics. This system of covariants, five of which are pure invariants, is derived in explicit form.

The methods of generation and proof of the completeness of the fundamental set are developed from the point of view emphasized in a paper on the formal modular invariant theory, by the present writer, in volume 17 of these *Transactions*.† These methods presuppose a knowledge of a fundamental system of formal seminvariants of the given ground form; but this seminvariant system has been given previously‡ for f_3 and the modulus 2, by Dickson.

1. RESUMÉ OF METHODS

We recapitulate in (a), (b), (c), (d) certain processes, previously given, which are apropos in the developments relating to f_3 . In (e), (f) novel principles are developed.

* Presented to the Society, April 28, 1917.

† The following papers by the present writer will be referred to by number:

I. *American Journal of Mathematics*, vol. 37 (1915), p. 73.

II. *Bulletin of the American Mathematical Society*, vol. 21 (1915), p. 167.

III. *These Transactions*, vol. 17 (1916), p. 545.

‡ Dickson, *Madison Colloquium Lectures* (1913), p. 53.

(a) *Transvection.* In I it was shown that transvection between a form, as f_3 , and the forms of the complete system of universal covariants of the group $A \pmod{p}$ yields numerous formal concomitants mod p . These universal covariants are*

$$(1) \quad L = x_1^p x_2 - x_1 x_2^p, \quad Q = (x_1^{p^2} x_2 - x_1 x_2^{p^2}) \div L.$$

(b) *Modular polars.* In II the following invariantive operators \pmod{p} were introduced in connection with a binary m -ic f_m :

$$E = x_1^p \frac{\partial}{\partial x_1} + x_2^p \frac{\partial}{\partial x_2}, \quad w = a_0^p \frac{\partial}{\partial a_0} + \cdots + a_m^p \frac{\partial}{\partial a_m}.$$

(c) *Concomitants of the first degree.* Every form of order > 3 is shown in III to be reducible modulo 2 in terms of first degree invariants and first degree covariants of orders 1, 2, and 3. A set of concomitants mod 2 of f_m , of degree 1, is the following:

$$(2) \quad K = a_1 + \cdots + a_{m-1}, \quad K_1 = (a_0 + K)x_1 + (K + a_m)x_2, \\ K_2 = a_0 x_1^2 + K x_1 x_2 + a_m x_2^2.$$

These three exist for all orders. If m is odd there exists a cubic covariant

$$(3) \quad K_{m3} = a_0 x_1^3 + I_1 x_1^2 x_2 + I_2 x_1 x_2^2 + a_m x_2^3 \quad (I_1 + I_2 = K).$$

(d) *Copied forms.* If f_m, g_n are two binary forms and σ is a system of modular concomitants of g_n , then a system for any covariant F_n of f_m , constructed on the model of σ , is a system of concomitants of f_m .

(e) *Hexadic scales.* There exist, in general, an infinite number of covariants mod p having one and the same seminvariant leading coefficient. Let $F_M = C_0 x_1^M + C_1 x_1^{M-1} x_2 + \cdots$ be any covariant modulo 2, of odd order M , of f_m , and construct concomitants of F_M on the models of K, K_1, K_2, K_{m3} (cf. (2), (3)). These copied forms are concomitants of f_m , viz.,

$$(4) \quad D = C_1 + \cdots + C_{M-1}, \quad D_1 = (C_0 + D)x_1 + (D + C_M)x_2, \\ D_2 = C_0 x_1^2 + D x_1 x_2 + C_M x_2^2, \\ F_{M3} = C_0 x_1^3 + J_1 x_1^2 x_2 + J_2 x_1 x_2^2 + C_M x_2^3 \quad (J_1 + J_2 = D).$$

LEMMA. Corresponding to any given cubic covariant mod 2 of f_m , as,

$$F_3 = C_0 x_1^3 + C_1 x_1^2 x_2 + C_2 x_1 x_2^2 + C_3 x_2^3,$$

there exists a definite cubic covariant Γ whose leading coefficient is the invariant

* Dickson, these Transactions, vol. 12 (1911), p. 75, and Madison Colloquium Lectures (1913), p. 33.

$C_1 + C_2$, viz.,

$$(5) \quad \Gamma = (C_1 + C_2)x_1^3 + (C_0 + C_1 + C_3)x_1^2x_2 \\ + (C_0 + C_2 + C_3)x_1x_2^2 + (C_1 + C_2)x_2^3.$$

To prove the covariance of Γ we assume that F_3 is a covariant, necessary and sufficient conditions for which are (1) homogeneity, (2) invariance under the permutational substitution $s = (a_0 a_m)(a_1 a_{m-1}) \cdots (x_1 x_2)$, and (3) invariance under the operation of transforming f_m by $T: x_1 \equiv x'_1 + x'_2, x_2 \equiv x'_2 \pmod{2}$. Let the increments of C_i ($i = 0, \dots, 3$), when f_m is transformed into f'_m by T , be δC_i ($i = 0, \dots, 3$) respectively. Then if F'_3 is the F_3 function constructed from f'_m ,

$$F'_3 \equiv C_0 x_1^3 + (C_0 + C_1 + \delta C_1)x_1^2x_2 + (C_0 + C_2 + \delta C_2)x_1x_2^2 \\ + (C_0 + C_1 + C_2 + C_3 + \delta C_1 + \delta C_2 + \delta C_3)x_2^3 \equiv F_3 \pmod{2},$$

whence follows

$$(6) \quad \delta C_0 \equiv 0, \quad \delta C_1 \equiv C_0, \quad \delta C_2 \equiv C_0, \quad \delta C_3 \equiv C_0 + C_1 + C_2 \pmod{2}.$$

Constructing Γ' and applying (6) we obtain immediately $\Gamma' \equiv \Gamma \pmod{2}$, which proves the lemma.

Application of this lemma to F_{M3} of (4) gives the covariant

$$(7) \quad \Gamma_M = Dx_1^3 + (C_0 + C_M + J_1)x_1^2x_2 + (C_0 + C_M + J_2)x_1x_2^2 + Dx_2^3.$$

Observe that J_1, J_2 are interchanged by the substitution s . Hence, since $J_1 + J_2 \equiv D$, the invariant leading coefficient of Γ_M can contain no term $a'_0 a'_1 a'_2 \cdots$ which is left unaltered by s . This is true of all invariants which lead cubic covariants, for, if the leading coefficient C_0 of F_3 is an invariant, we have $C_0 \equiv C_3$, and $\delta C_3 \equiv 0$. Then (6) gives

$$C_0 \equiv C_1 + C_2 \pmod{2},$$

and, as the covariance of F_3 requires that C_1 and C_2 be interchanged by s , any term of C_1 which is left unaltered by s occurs also in C_2 and therefore has a zero coefficient, modulo 2, in the sum C_0 .

We shall designate the six interrelated concomitants $F_M, F_{M3}, \Gamma_M, D, D_1, D_2$ as the *hexadic scale** for the covariant F_M of odd order M . Every covariant of odd order furnishes such a scale of concomitants.

(f) *Tetradic scales*. If F_M is a covariant mod 2 of even order M no cubic covariant corresponding to F_{M3} exists, but we have the interrelated forms F_M, D, D_1, D_2 , which, accordingly, will be called a *tetradic scale* for F_M .

*The term scale was used by Sylvester in the sense of a fundamental system, but this designation has become practically obsolete.

Note that the reducibility of F_M does not imply the reducibility of the forms in the scale for F_M , but the covariant Γ_M in any hexadic scale is reducible in terms of other concomitants in the scale, as follows:

$$(8) \quad \Gamma_M \equiv QD_1 + F_{M3} \pmod{2}.$$

2. THE GENERAL SEMINVARIANT OF f_3

If

$$f_3 = a_0 x_1^3 + a_1 x_1^2 x_2 + a_2 x_1 x_2^2 + a_3 x_2^3,$$

the fundamental system of formal seminvariants modulo 2, given by Dickson, is composed of

$$(9) \quad \begin{aligned} a_0, \quad K &= a_1 + a_2, \quad \delta_{00} = (a_0 + K + a_3) a_3, \\ \Delta &= a_0 a_3 + a_1 a_2, \quad \beta = a_0 a_1 + a_1^2. \end{aligned}$$

With these may be associated the following invariants:*

$$(10) \quad \begin{aligned} K, \quad \Delta, \quad I &= a_0^2 + a_0 K + \delta_{00}, \quad k = a_0 \delta_{00}, \\ g &= \beta^2 + \beta(\Delta + K^2) + (\Delta + \delta_{00})(\beta + a_0 K + K^2). \end{aligned}$$

We have noticed previously, in III, the following syzygies connecting seminvariants. They result immediately from (9) and (10).

$$(11) \quad \left. \begin{aligned} a_0^3 + a_0^2 K + a_0 I + k &\equiv 0, \\ \beta^2 + \beta(a_0^2 + a_0 K + I + K^2) \\ &+ (a_0^2 + a_0 K + I + \Delta)(a_0 K + K^2) + g \equiv 0 \end{aligned} \right\} \pmod{2}.$$

It will be convenient, in this paper,† to abbreviate the invariant $g + I\Delta + \Delta^2$ as g_1 . Thus,

$$(11_1) \quad g \equiv g_1 + I\Delta + \Delta^2 \pmod{2}.$$

Any seminvariant ϕ of f_3 , being a polynomial in the seminvariants (9), is a polynomial in a_0, K, I, β, Δ ,

$$\phi = \phi(a_0, \beta, K, \Delta, I).$$

Hence when the congruences (11) are used as reducing moduli with respect to powers of a_0 and of β , ϕ can be reduced to the form

$$(12) \quad \phi = J_0 + J_1 a_0 + J_2 a_0^2 + (\Gamma_0 + \Gamma_1 a_0 + \Gamma_2 a_0^2) \beta,$$

where J_i, Γ_i ($i = 0, 1, 2$) are invariants expressed as polynomials in K, Δ ,

* Cf. Dickson, loc. cit.; and III, pp. 554, 555.

† The advantage of this change is that g contains terms which are symmetrical under s (§ 1) and hence g cannot be the leading coefficient of any covariant of odd order, whereas we shall determine a cubic covariant led by g_1 .

k, I, g_1 . Formula (12) is therefore the general form of a seminvariant leader of a formal covariant of f_3 . Another form of ϕ which we shall employ is obtained by introducing B into (12) through the defining relation $B \equiv \beta + \Delta$. Thus,

$$(13) \quad \phi \equiv R_0 + R_1 a_0 + R_2 a_0^2 + (S_0 + S_1 a_0 + S_2 a_0^2) B,$$

where R_i, S_i ($i = 0, \dots, 2$), like J_i, Γ_i ($i = 0, \dots, 2$), are polynomials in K, Δ, k, I, g_1 , with numerical coefficients.

3. CONSTRUCTION OF COVARIANTS OF f_3

The hexadic scale for the quantic f_3 itself, composes a system of first degree concomitants. The forms in this scale are f_3, f_3, G, K, K_1, K_2 , where (cf. III, p. 555)

$$(14) \quad \begin{aligned} G &\equiv Kx_1^3 + (a_0 + a_1 + a_3)x_1^2x_2 + (a_0 + a_2 + a_3)x_1x_2^2 + Kx_2^3 \\ &\equiv QK_1 + f_3 \pmod{2}. \end{aligned}$$

The hessian of f_3 is both an algebraical and a formal modular covariant, viz.,

$$H \equiv 3x_1^2 + \Delta x_1x_2 + (a_1a_3 + a_2^2)x_2^2 \quad (s = a_0a_2 + a_1^2).$$

The tetradic scale for QH consists of QH, H, G_1 , and Δ , where

$$G_1 \equiv (s + \Delta)x_1 + (\Delta + a_1a_3 + a_2^2)x_2 \quad [\equiv (H, L)^2].$$

A quadratic covariant led by β is

$$(15) \quad t \equiv KK_2 + H \equiv \beta x_1^2 + (\Delta + K^2)x_1x_2 + (a_2a_3 + a_2^2)x_2^2,$$

and in the tetradic scale for Qt occurs

$$(16) \quad \begin{aligned} t_1 &\equiv (\beta + \Delta + K^2)x_1 + (\Delta + K^2 + a_2a_3 + a_2^2)x_2 \\ &\equiv G_1 + KK_1 \pmod{2}. \end{aligned}$$

Apply to the forms just derived the modular operator

$$w \equiv a_0^2 \frac{\partial}{\partial a_0} + a_1^2 \frac{\partial}{\partial a_1} + a_2^2 \frac{\partial}{\partial a_2} + a_3^2 \frac{\partial}{\partial a_3}.$$

Thus we get

$$(17) \quad \begin{aligned} C_1 &\equiv wK_1 \equiv (a_0^2 + K^2)x_1 + (K^2 + a_3^2)x_2, \\ C_2 &\equiv wK_2 \equiv a_0^2x_1^2 + K^2x_1x_2 + a_3^2x_2^2 \equiv K_1^2 + K^2Q \pmod{2}, \\ P &\equiv wf_3 \equiv a_0^2x_1^3 + a_1^2x_1^2x_2 + a_2^2x_1x_2^2 + a_3^2x_2^3. \end{aligned}$$

In order to perform the reductions necessary to isolate the complete system sought we shall need, among other forms, quadratic covariants led by the

seminvariants $a_0 \beta$ and $a_0^2 \beta$ respectively, and cubic covariants led by B , $a_0 B$, and $a_0^2 B$. These are derived by forming certain scales of concomitants, as follows:

Construct the hexadic scale for the quintic covariant $f_3 t$, and we have, neglecting reducible forms,

$$\begin{aligned}
 F_{53} &\equiv a_0 \beta x_1^3 + (a_0^2 a_3 + a_1 a_2^2 + a_0 a_1 a_3 + a_0 a_2 a_3) x_1^2 x_2 \\
 &\quad + (a_0 a_3^2 + a_1^2 a_2 + a_0 a_2 a_3 + a_0 a_1 a_3) x_1 x_2^2 + (a_2 a_3^2 + a_2^2 a_3) x_2^3, \\
 (18) \quad D_2 &\equiv a_0 \beta x_1^2 + (k + K\Delta) x_1 x_2 + (a_2 a_3^2 + a_2^2 a_3) x_2^2, \\
 D_1 &\equiv (a_0 \beta + k + K\Delta) x_1 + (k + K\Delta + a_2 a_3^2 + a_2^2 a_3) x_2.
 \end{aligned}$$

Next we form the hexadic scale for the quintic tP , giving

$$\begin{aligned}
 F'_{53} &\equiv a_0^2 \beta x_1^3 + (a_0^3 a_3 + a_0^2 a_1 a_2 + a_0^2 a_2 a_3 + a_0^2 a_1^2 + a_0 a_1^2 a_3 \\
 &\quad + a_0 a_1 a_2^2 + a_1^3 a_2) x_1^2 x_2 + (a_1 a_2^3 + a_0 a_2^2 a_3 + a_1^2 a_2 a_3 + a_2^3 a_3 \\
 (19) \quad &\quad + a_0 a_1 a_3^2 + a_1 a_2 a_3^2 + a_2^2 a_3^2 + a_0 a_3^3) x_1 x_2^2 + (a_2 a_3^3 + a_2^2 a_3^2) x_2^3, \\
 D'_2 &\equiv a_0^2 \beta x_1^2 + (I\Delta + \Delta^2 + g) x_1 x_2 + (a_2 a_3^3 + a_2^2 a_3^2) x_2^2, \\
 D'_1 &\equiv (a_0^2 \beta + I\Delta + \Delta^2 + g) x_1 + (I\Delta + \Delta^2 + g + a_2 a_3^3 + a_2^2 a_3^2) x_2.
 \end{aligned}$$

Again, the scale for the quintic Ql , where l is the cubic

$$\begin{aligned}
 (20) \quad l &\equiv QG_1 + Kf_3 \equiv Bx_1^3 + (a_0 a_2 + a_1 a_2 + a_1 a_3 + a_2^2) x_1^2 x_2 \\
 &\quad + (a_0 a_2 + a_1 a_2 + a_1 a_3 + a_1^2) x_1 x_2^2 + (\Delta + a_2 a_3 + a_2^2) x_2^3,
 \end{aligned}$$

furnishes the covariants

$$\begin{aligned}
 (21) \quad l_2 &\equiv Bx_1^2 + K^2 x_1 x_2 + (\Delta + a_2 a_3 + a_2^2) x_2^2, \\
 t_1 &\equiv (B + K^2) x_1 + (K^2 + \Delta + a_2 a_3 + a_2^2) x_2,
 \end{aligned}$$

and it is now evident that cubic covariants, say F and F' , led by the respective seminvariants $a_0 B$, $a_0^2 B$ may also be constructed. For F is the cubic covariant (the one not led by an invariant) in the hexadic scale for the quintic $f_3 l_2$ and F' is the corresponding cubic in the scale for Pl_2 . We also write F and F' explicitly, but they will be proved to be reducible (cf. (26)):

$$\begin{aligned}
 F &\equiv a_0 Bx_1^3 + (a_0^2 a_3 + a_0 a_1 a_3) x_1^2 x_2 + (a_0 a_3^2 + a_0 a_2 a_3) x_1 x_2^2 \\
 &\quad + (a_0 a_3^2 + a_1 a_2 a_3 + a_2 a_3^2 + a_2^2 a_3) x_2^3, \\
 (22) \quad F' &\equiv a_0^2 Bx_1^3 + (a_0^3 a_3 + a_0^2 a_1^2 + a_0^2 a_1 a_2 + a_0^2 a_2 a_3 + a_0 a_1^2 a_3 + a_0 a_2^2 a_3 \\
 &\quad + a_0 a_1 a_2^2 + a_0 a_1^3 + a_1^3 a_2 + a_1 a_2^3) x_1^2 x_2 + (a_0 a_1^2 a_3 + a_0 a_2^2 a_3 \\
 &\quad + a_0 a_1 a_3^2 + a_0 a_3^3 + a_1^2 a_2 a_3 + a_2^2 a_3 + a_1 a_2 a_3^2 + a_2^2 a_3^2 \\
 &\quad + a_1^3 a_2 + a_1 a_2^2) x_1 x_2^2 + (a_0 a_3^3 + a_1 a_2 a_3^2 + a_2 a_3^3 + a_2^2 a_3^2) x_2^3.
 \end{aligned}$$

4. COVARIANTS OF f_3 , WHOSE LEADING COEFFICIENTS ARE INVARIANTS

Reduction methods to be employed in the next section require an explicit knowledge of all covariants which have as leading coefficients pure invariants which are polynomials in the invariants K, Δ, k, I, g_1 , homogeneous as to a_0, \dots, a_3 . We can write the general polynomial in question in the form

$$\Phi = f(I, \Delta) + K\psi_1 + k\psi_2 + g_1\psi_3,$$

where $f(I, \Delta)$ is a polynomial in I and Δ only, and ψ_i ($i = 1, 2, 3$) are polynomials involving, in general, all five invariants.

LEMMA. *No covariant of odd order exists having $f(I, \Delta)$ as a leading coefficient.*

In proof of this we show that every polynomial in I and Δ alone, which is homogeneous in a_0, \dots, a_3 , necessarily has a term which is left unaltered by the substitution $s = (a_0 a_3)(a_1 a_2)$. In fact the only symmetrical term in I^p is $a_0^p a_3^p$,* while all terms in Δ^q are symmetrical under s . If $I^p \Delta^q$ is the term containing the highest power of Δ in $f(I, \Delta)$, then the term

$$\tau = a_0^p a_1^q a_2^q a_3^p$$

certainly occurs in f with a numerical coefficient which is $\not\equiv 0 \pmod{2}$. But, as shown in § 1 (e), no covariant of odd order can be led by an invariant containing a term τ unaltered by s . This proves the lemma.

LEMMA. *There exists both a quadratic and a cubic covariant (but no linear covariant) led by each one of the three invariants K, k, g_1 .*

The quadratic covariants are the products of the respective invariants K, k, g_1 by Q .

The cubic led by K is the covariant G in (14).

A cubic covariant led by k is found by constructing the following polynomial in concomitants derived in § 3:

$$T = QD_1 + F_{53} + \Delta G;$$

$$\begin{aligned} (23) \quad T = & kx_1^3 + (a_0^2 a_1 + a_0 a_1^2 + a_2 a_3^2 + a_2^2 a_3 + a_1 a_2^2 + a_0 a_2 a_3 \\ & + a_0 a_1 a_2 + a_1 a_2 a_3 + a_1^2 a_2 + a_0 a_3^2) x_1^2 x_2 + (a_2 a_3^2 + a_2^2 a_3 \\ & + a_0^2 a_1 + a_0 a_1^2 + a_1^2 a_2 + a_0 a_1 a_3 + a_1 a_2 a_3 + a_0 a_1 a_2 + a_1 a_2^2 \\ & + a_0^2 a_3) x_1 x_2^2 + kx_2^3. \end{aligned}$$

Finally, a cubic covariant whose leader is the invariant $g_1 = I\Delta + \Delta^2 + g$ is the covariant led by an invariant, belonging to the hexadic scale for tP .

* The expression z_4^n is the only term in the expansion of $(z_1 + z_2 + \dots + z_7)^n$ which has an odd coefficient and is left unaltered by $r = (z_1 z_7)(z_2 z_6)(z_3 z_5)(z_4)$.

By (8) this covariant, say E , is reducible as follows (cf. (19)):

$$(24) \quad E \equiv QD'_1 + F'_{53} \pmod{2}.$$

Combining the preceding results in this section we conclude as in the following

LEMMA. *The most general form for a pure invariant leading coefficient of a covariant of f_3 , of odd order, is*

$$S = K\psi_1 + k\psi_2 + g_1\psi_3,$$

and the following quantic is a cubic covariant led by S :

$$(25) \quad \Psi = \psi_1 G + \psi_2 T + \psi_3 E.$$

5. THE FUNDAMENTAL SYSTEM OF COVARIANTS OF f_3

Every covariant mod 2 of even order, of f_3 , is of the form $R = \phi x_1^{2h} + \dots$, where the leading coefficient is the ϕ function of (12). We shall construct, as a rational integral function of the concomitants which were derived in the two preceding sections, another covariant C whose seminvariant leading coefficient is ϕ . Then $R - C$, although not vanishing in general, will always be congruent to the product of a covariant C' of odd order $2h - 3$ by L . This consideration, the method of which is due to Dickson, evidently furnishes a general reduction method,* since the same process can be applied to C' and to the covariants analogous to C' , in succession.

For example, if we subtract $F_{53} + \Delta f_3$ from F (cf. (18) and (22)), and reduce the factor C' in the remainder, we get the first relation below. The second relation is given by performing similar operations in connection with the covariant F' of (22),

$$(26) \quad \begin{aligned} F &\equiv F_{53} + \Delta f_3 + LK\Delta, \\ F' &\equiv F'_{53} + \Delta P + LK^2\Delta \pmod{2}. \end{aligned}$$

In the general case, when $R = \phi x_1^{2h} + \dots$ ($h \geq 1$), we find

$$(27) \quad \begin{aligned} R &\equiv J_0 Q^h + J_1 K_2 Q^{h-1} + J_2 C_2 Q^{h-1} + \Gamma_0 t Q^{h-1} + \Gamma_1 D_2 Q^{h-1} \\ &\quad + \Gamma_2 D'_2 Q^{h-1} + LC' \pmod{2}, \end{aligned}$$

where $C' \equiv 0$ if $h = 1$, and C' is a covariant of odd order $2h - 3$ if $h > 1$.

Next, when R is of odd order $2h + 1 > 1$, namely $R = \phi x_1^{2h+1} + \dots$, we deduce by use of the form (13) of ϕ , involving B , and the last lemma in Section 4,

$$(28) \quad R \equiv (\Psi + R_1 f_3 + R_2 P + S_0 l + S_1 F + S_2 F') Q^{h-1} + LC' \pmod{2},$$

* Dickson, *Proof of the finiteness of modular covariants*, these Transactions, vol. 14 (1913), p. 299.

where, if $h = 1$, C' is an invariant, and, if $h > 1$, C' is a covariant of even order $2h - 2$.

After we have applied these processes successively to the covariants C' , we shall have reduced all covariants of order $\neq 1$, excepting the irreducible concomitants of orders 0, 1, 2, 3 in terms of which the covariants in formulas (27), (28) are explicitly constructed, as in formulas (14) to (26).

Consider next the covariants which are linear in x_1, x_2 , all being of the form

$$\lambda = \phi x_1 + \phi_1 x_2.$$

From (12) we have, identically,

$$(29) \quad \phi = N + J_1(a_0 + K) + J_2(a_0^2 + K^2) + \Gamma_0(\beta + \Delta + K^2) \\ + \Gamma_1(a_0\beta + k + K\Delta) + \Gamma_2(a_0^2\beta + g_1) \pmod{2},$$

in which N is an invariant (cf. (11₁));

$$N = J_0 + J_1 K + J_2 K^2 + \Gamma_0 \Delta + \Gamma_0 K^2 + \Gamma_1 k + \Gamma_1 K\Delta + \Gamma_2 g_1 \pmod{2}.$$

We can construct a linear covariant led by each parenthesis in (29) (cf. (30)). No linear covariant is led by an invariant. Hence, assuming λ to be a covariant, $N \equiv 0 \pmod{2}$. Making use of the linear covariants in the various scales of concomitants in Section 3, we have,

$$(30) \quad \lambda = J_1 K_1 + J_2 C_1 + \Gamma_0 t_1 + \Gamma_1 D_1 + \Gamma_2 D'_1 \pmod{2}.$$

Hence all linear covariants are reducible in terms of the invariants K, Δ, k, I, g_1 and the five covariants K_1, C_1, t_1, D_1, D'_1 .

Giving attention, now, to the covariants entering the formulas (27), (28), (30), expressed as rational integral functions of other covariants by the explicit formulas given in Sections 3 and 4, we summarize our conclusions, in the following

THEOREM. *A fundamental system of formal covariants modulo 2 of the binary cubic quantic f_3 is composed of twenty forms, as follows: Five invariants K, Δ, k, I, g_1 ; five linear covariants K_1, C_1, t_1, D_1, D'_1 ; four quadratic covariants K_2, t, D_2, D'_2 ; and four cubic covariants f_3, P, F_{53}, F'_{53} , together with the two universal covariants L, Q .*

6. SYZYGIES

The syzygies connecting the members of this fundamental system are legionary. In the paper quoted as III above I gave a theorem which establishes the existence and furnishes a method of construction of an infinitude of syzygies, although this theorem does not directly furnish all such relations. Each identity (11) connecting the fundamental seminvariants of f_3 furnishes a

syzygy, for if we substitute in one of these, for each seminvariant, a covariant which the latter leads, paying attention to considerations of homogeneity, we get a reducible covariant which is congruent to L times a covariant C' , and reduction of C' leads to a syzygy. The Σ_1 below was constructed from the first relation in (11); Σ_2 by the before-mentioned theorem:

$$\begin{aligned} \Sigma_1 &= K_1^2 K_2 + K^2 K_2 Q + KK_1^2 Q + K^3 Q^2 + IK_2 Q \\ &\quad + kQ^2 + K^2 K_1 L + IK_1 L \equiv 0 \pmod{2}, \\ (31) \quad \Sigma_2 &= tf_3 + K_2 t_1 Q + KK_1 K_2 Q + KK_2 f_3 + \Delta f_3 Q + D_2 L \\ &\quad + KK_1^2 L + K^3 LQ + \Delta KLQ \equiv 0 \pmod{2}. \end{aligned}$$

Since a syzygy is a polynomial in the fundamental concomitants we can prove that all are expressible in terms of a finite set of irreducible ones, $\Sigma_1, \dots, \Sigma_r$, by applying Hilbert's theorem, replacing the equations customarily understood in this theorem by identical congruences modulo 2. That is, any syzygy Σ may be put in the form

$$\Sigma \equiv \sigma_1 \Sigma_1 + \sigma_2 \Sigma_2 + \dots + \sigma_r \Sigma_r \pmod{2},$$

where $\sigma_1, \dots, \sigma_r$, being polynomials in the concomitants, are themselves concomitants of f_3 .

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